SIM Global Education

SIM MODULE BOOK

BUSINESS MATHEMATICS

Module Book
BUSINESS MATHEMATICS

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INTRODUCTION

Content

This module introduces and crystallizes essential mathematical concepts and techniques for a firm and creative understanding of foundation mathematics required for business-related studies. The importance of the core concepts and techniques are illustrated and developed into a generic mature approach to mathematics and creative problem solving.

Module Aims

The aims of this module are to:

1. Equip participants with a sound understanding and appreciation of important central ideas in mathematics.

2. Inculcate participants with a healthy and mature approach towards problem solving.

Learning Outcomes

On completion of this module, a participant will typically be able to:

1. Acquire a detailed knowledge and understanding of essential mathematical principles.

2. Communicate knowledge and understanding of mathematics through mathematical expressions, charts, graphs, diagrams and prose.

3. Develop and demonstrate transferable skills in mathematical modelling, analytical reasoning and problem solving.

Delivery of Module and Lesson Plan			
S/N.	**Topic**	**Learning Outcomes** At the completion of this topic, participants will be able to:	**Text**
1.	Fundamentals of Algebra (I)	1. Understand the concept of real numbers and their representation on a number line. 2. Simplify exponential, polynomial and rational expressions. 3. Simplify rational exponents and radicals.	Business Mathematics Module Book Session 1
2.	Fundamentals of Algebra (II)	1. Solve linear and quadratic equations. 2. Solve linear, quadratic, rational and absolute value inequalities. 3. Find terms and compute partial sums of an arithmetic or geometric progression.	Business Mathematics Module Book Session 2
3.	Functions and their graphs (I)	1. Determine the gradient and equation of a straight line in the x-y plane. 2. Understand the concept of a function and determine its domain and range. 3. Compute the sum, difference, product, quotient and composition of functions.	Business Mathematics Module Book Session 3
4.	Functions and their graphs (II)	1. Sketch graphs of linear and quadratic functions. 2. Apply linear functions to linear depreciation, cost, revenue and profit functions and break-even analysis. 3. Apply quadratic functions to demand and supply curves, market equilibrium quantities. 4. Understand the use of functions in mathematical modelling.	Business Mathematics Module Book Session 4
5.	Exponential and logarithmic functions	1. Understand the relation between the exponential and logarithm functions. 2. Apply the laws of exponential and logarithm functions to solve equations. 3. Sketch the graphs of exponential and logarithm functions. 4. Solve problems related to exponential growth, exponential decay and logistic growth models.	Business Mathematics Module Book Session 5
6.	Mathematics of finance	1. Solve problems involving simple interest, compound interest and continuous compounding of interest. 2. Compute the effective rate of interest.	Business Mathematics Module Book

Delivery of Module and Lesson Plan			
S/N.	**Topic**	**Learning Outcomes** At the completion of this topic, participants will be able to:	**Text**
		3. Compute the future value and present value of an annuity. 4. Solve problems involving the amortization of a loan and sinking fund.	Session 6
7.	Calculus – Derivatives and its application I	1. Compute limits of simple functions. 2. Understand the concept of continuous functions and their properties. 3. Compute the slope of a tangent line using first principles. 4. Use the basic rules of differentiation to compute derivatives of polynomials.	Business Mathematics Module Book Session 7
8.	Calculus – Derivatives and its application II	1. Use the product rule, quotient rule, chain rule and general power rule to compute derivatives of complex functions. 2. Compute derivatives of exponential and logarithmic functions. 3. Understand and derive marginal cost, revenue and profit functions. 4. Find the relative extrema of a function.	Business Mathematics Module Book Session 8
9.	Calculus – Derivatives and its application III	1. Find higher order derivatives of a function. 2. Apply the second derivative to determine the intervals of concavity of a function and hence establish the nature of a relative extrema. 3. Solve optimization problems using derivatives in business and economics.	Business Mathematics Module Book Session 9
10.	Calculus – Integration and its application I	1. Find indefinite integrals of polynomial and exponential functions. 2. Solve simple initial value problems. 3. Compute integrals by using substitution.	Business Mathematics Module Book Session 10
11.	Calculus – Integration and its application II	1. Relate definite integrals to areas enclosed by graphs and lines. 2. Apply the Fundamental Theorem of Calculus to compute definite integrals. 3. Apply definite integrals to compute nett change of a quantity and average value of a function.	Business Mathematics Module Book Session 11

Delivery of Module and Lesson Plan			
S/N.	**Topic**	**Learning Outcomes** At the completion of this topic, participants will be able to:	**Text**
12.	Calculus – Integration and its application III	1. Compute areas between two curves. 2. Apply the areas between two curves to compute consumers' and producers' surplus. 3. Apply definite integrals to compute the future and present value of an income stream and the future and present value of an annuity.	Business Mathematics Module Book Session 12

Teaching and Learning Methods
Participants will learn through a combination of lectures and practical activities. Participants will be expected to learn independently by carrying out reading and directed study beyond that available within taught classes.

Indicative Readings	
Textbooks required	Chue Kah Loong, Business Mathematics Module Book, SIM Global Education, 2017

Assessment/coursework

All assessments must comply with the SIM Rules and Regulations. To satisfy module requirements, students must:

1) Satisfactorily complete and present on due dates their completed assignment. A penalty of 20% of the total marks will be imposed for late submission. A submission later than 1 calendar day past deadline will receive a zero mark.
2) Complete all assignments and the final examination in a satisfactory manner.
3) Reference all their work and observe SIM's policy on plagiarism. Students found guilty of plagiarism will be dealt with severely.
4) Adopt either the Harvard or APA (American Psychological Association) Referencing Styles.
5) Spend at least 100 hours (including class attendance and assignments) on the module in order to fare reasonably.

Specific for this module are the following requirements:

Weighting between components A and B - A: 70% **B:** 30%	
Element Description	**% of Assessment**
Component A (Controlled Conditions)	
Examination (180 minutes)	70%
Component B (CA: Continuous Assessment)	
1. CA 1 2. CA 2	30%
Total (Component A+B)	100%

BUSINESS MATHEMATICS

SESSION 1

FUNDAMENTALS OF ALGEBRA I

At the end of the session, students should be able to:

1. Understand the concept of real numbers and their representation on a number line.
2. Simplify exponential, polynomial and rational expressions.
3. Simplify rational exponents and radicals.

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1. Introduction

This session is the first part of a brief review of algebra concepts that you will use in this course. It covers the representation of real numbers on a number line and operations on algebraic expressions such as polynomials, exponents and radicals.

Real numbers are used in everyday life. Every time we use a laptop, send a message over the phone or watch a video, we use devices that rely on real numbers to operate. Real numbers are also used to describe the natural world, communicate information and to model problems in reality.

Whereas real numbers deal in specific quantities, algebra deals with unknown quantities by representing these unknown values with letters called variables. By combining various operations on these variables, algebraic expressions are formed. These expressions allow us to describe general problems and form mathematical relationships between quantities that may vary. Polynomials, rational and exponential expressions are all various types of algebraic expressions.

2. Real Numbers

2.1 The Set of Real Numbers

Real numbers can be defined by a construction approach. We first begin with the set of natural numbers (also called counting numbers).

$$\mathbb{N} = \{1, 2, 3, ...\}$$

The number 0 is joined to the natural numbers to form whole numbers.

$$\mathbb{W} = \{0, 1, 2, 3, ...\}$$

The set of integers are then formed by adding the negatives of the natural numbers.

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Next, we consider the set of rational numbers, which are numbers of the form $\frac{p}{q}$, i.e. numbers that can be written as a fraction.

$$\mathbb{Q} = \left\{ \frac{p}{q} \middle| \; p \text{ and } q \text{ are integers, } q \neq 0 \right\}$$

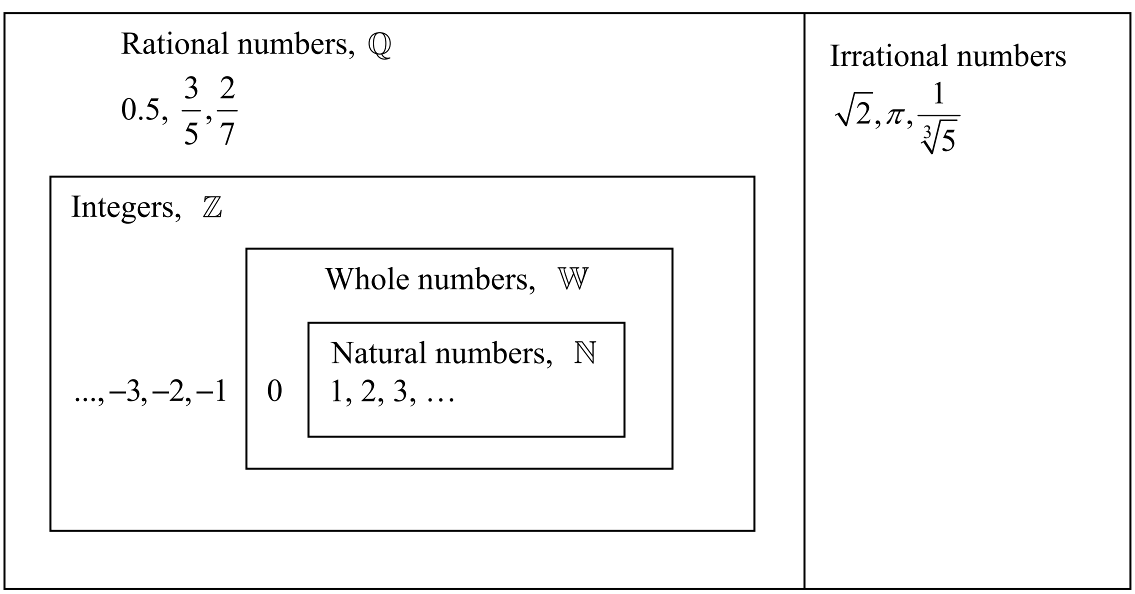
The set of integers is contained in the set of rational numbers as any integer can be written in the form $\frac{p}{1}$, e.g. $5 = \frac{5}{1}$. However, not all rational numbers can be written as integers, e.g. $\frac{5}{7}$. Note also that the denominator of a rational number cannot be zero. Division by 0 is undefined, e.g. $\frac{4}{0}$ is undefined.

Finally, we consider the set of irrational numbers, i.e. numbers that cannot be expressed as a fraction. Examples of irrational numbers are $\sqrt{2}, \sqrt{5}, \pi$.

The set of real numbers, $\mathbb{R}$, contains all rational numbers and irrational numbers. The relation between the various sets of numbers is illustrated in Figure 1.1 below.

Figure 1.1: The real number system

Real numbers, $\mathbb{R}$



2.2 Representing Real Numbers as Decimals

Every real number can be expressed as a decimal. If the number is a rational number, the decimal is either a repeating decimal or will terminate. If the number is an irrational number, the decimal will not repeat nor terminate. For example,

$$\frac{1}{3} = 0.333333\text{, a repeating decimal that can be written as } 0.\dot{3}$$

$$\frac{5}{8} = 0.625\text{, a terminating decimal}$$

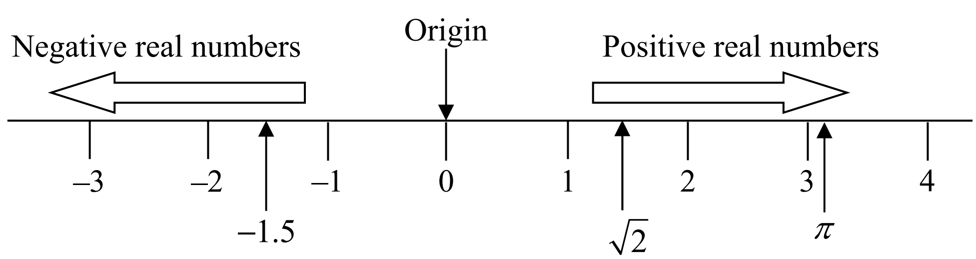
$$\sqrt{2} = 1.41421356...\text{, a decimal that is neither repeating nor terminating}$$

2.3 Representing Real Numbers on a Number Line

A number line allows us to display numbers and their relationships visually. As a line contains infinitely many points, each point can be associated with a unique real number. We construct a number line by first drawing a horizontal line and arbitrarily selecting a point to represent the number 0. This point is also known as the origin. Choose a point to the right of the origin to represent the number 1. This determines the scale for the number line.

A positive real number, x, is represented by a point x units to the right of 0. A negative real number, $-x$, is represented by a point x units to the left of 0. In this manner, each real number can be represented by exactly one point on the number line and vice versa. A real number x is said to be greater than another real number y, i.e. $x > y$, if the point y is to the right of the point x. Figure 1.2 shows an example of a number line.

Figure 1.2: A real number line with the origin



2.4 Operations with Real Numbers

Two real numbers may be combined to obtain another real number through the operations of addition and multiplication. Addition of two numbers a and b allows us to compute their sum, denoted by $a + b$. Multiplication of two numbers a and b allows us to compute their product, denoted by $a \cdot b$ or more commonly ab. The two operations are subject to the rules operation given in Table 1.1.

Table 1.1: Rules of operation for real numbers

No.	Rule	Example
1.	$a + b = b + a$ [commutative law of addition]	$2 + 5 = 5 + 2$
2.	$a + (b + c) = (a + b) + c$ [associative law of addition]	$2 + (5 + 3) = (2 + 5) + 3$
3.	$ab = ba$ [commutative law of multiplication]	$(2)(5) = (5)(2)$
4.	$a(bc) = (ab)c$ [associative law of multiplication]	$(2)(5\cdot3) = (2\cdot5)(3)$
5.	$a(b + c) = ab + ac$ [distributive law]	$2(5 + 3) = (2)(5) + (2)(3)$

The operation of subtraction is defined in terms of addition, where $-b$ is known as the additive inverse of b. Thus $a + (-b)$ is more commonly written as $a - b$. Some of the properties for negative numbers are given in Table 1.2.

Table 1.2: Rules of operation for negative numbers

No.	Rule	Example
1.	$a + (-a) = 0$	$3 + (-3) = 0$
2.	$-(-a) = a$	$-(-3) = 3$
3.	$(-a)b = a(-b) = -ab$	$(-3)4 = 3(-4) = -(3)(4)$
4.	$(-a)(-b) = ab$	$(-3)(-4) = (3)(4)$
5.	$(-1)(a) = -a$	$(-1)(3) = -3$

The operation of division is defined in terms of multiplication. In this operation, $\frac{1}{b}$ is known as the multiplicative inverse of b. Division by zero is not defined, i.e. $b \neq 0$. Hence zero does not have a multiplicative inverse. In all other cases, $a \div b$ can be written as $a\left(\frac{1}{b}\right)$ or $\frac{a}{b}$.

The fraction $\frac{a}{b}$ is called a quotient and we can perform operations of addition, subtraction, multiplication and division on them. Some of these properties are given in Table 1.3.

Table 1.3: Rules of operation for quotients

No.	Rule	Example
1.	$\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$	$\frac{3}{5} = \frac{9}{15} \Leftrightarrow (3)(15) = (5)(9)$

2.	$\frac{ca}{cb}=\frac{a}{b}$	$\frac{(4)(3)}{(4)(5)}=\frac{3}{5}$
3.	$\frac{-a}{b}=\frac{a}{-b}=-\frac{a}{b}$	$\frac{-3}{4}=\frac{3}{-4}=-\frac{3}{4}$
4.	$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$	$\frac{3}{4}\cdot\frac{5}{7}=\frac{(3)(5)}{(4)(7)}$
5.	$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$	$\frac{3}{4}\div\frac{5}{7}=\frac{3}{4}\cdot\frac{7}{5}=\frac{(3)(7)}{(4)(5)}$
6.	$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$	$\frac{3}{4}+\frac{5}{7}=\frac{(3)(7)+(5)(4)}{(4)(7)}=\frac{41}{28}$
7.	$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$	$\frac{3}{4}-\frac{5}{7}=\frac{(3)(7)-(5)(4)}{(4)(7)}=\frac{1}{28}$

3. Exponential expressions

If a is any real number and k is a natural number, then the exponential expression a^k is defined as the number

$$a^k=\underbrace{a\cdot a\cdot a\cdot\ldots\cdot a}_{k\text{ times}}$$

The real number a is called the base and the natural number k is called the exponent. We read a^k as a to the power of k.

3.1 Operations with exponents

Two exponential expressions with the same base can be combined by multiplication. Consider the following product and quotient of two exponents.

$$2^3\cdot2^5=(2\cdot2\cdot2)\cdot(2\cdot2\cdot2\cdot2\cdot2)=2^8$$

$$\frac{2^5}{2^3}=\frac{2\cdot2\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}}=2^2$$

The above properties can be generalized to the product and quotient rule for exponents.

$$a^m\cdot a^n=a^{m+n}$$

$$\frac{a^m}{a^n}=a^{m-n}$$

We can extend these properties to derive other rules of exponents. These exponent rules are summarised and illustrated in Table 1.4.

Table 1.4: Rules for exponents

No.	Property	Example
1.	$a^m \cdot a^n = a^{m+n}$	$3^5 \cdot 3^2 = (3\cdot3\cdot3\cdot3\cdot3)\cdot(3\cdot3) = 3^7$
2.	$\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{3^5}{3^2} = \dfrac{3\cdot3\cdot3\cdot\cancel{3}\cdot\cancel{3}}{\cancel{3}\cdot\cancel{3}} = 3^3$
3.	$\left(a^m\right)^n = a^{mn}$	$\left(3^5\right)^4 = 3^5\cdot3^5\cdot3^5\cdot3^5 = 3^{20}$
4.	$a^0 = 1$	$\dfrac{3^5}{3^5} = 3^{5-5} = 3^0$ and $\dfrac{3^5}{3^5} = 1$. Hence $3^0 = 1$.
5.	$a^{-n} = \dfrac{1}{a^n}$	$\dfrac{3^2}{3^4} = 3^{2-4} = 3^{-2}$ and $\dfrac{3^2}{3^4} = \dfrac{\cancel{3}\cdot\cancel{3}}{3\cdot3\cdot\cancel{3}\cdot\cancel{3}} = \dfrac{1}{3^2}$. Hence $3^{-2} = \dfrac{1}{3^2}$.
6.	$(ab)^n = a^n b^n$	$(3\cdot5)^2 = 3^2\cdot5^2$
7.	$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\left(\dfrac{3}{5}\right)^2 = \dfrac{3^2}{5^2}$

3.2 Simplifying exponent expressions

The next two examples demonstrate the use of the rules to simplify exponent expressions.

Example 1.1 Simplify the following expressions, writing your answers using positive exponents only.

(a) $\left(3x^7\right)\left(11x^6\right)$ (b) $\dfrac{72y^8}{3y^9}$ (c) $\left(2z^{-2}\right)^{-3}(z)^{-6}$

Solution

(a)
$$\begin{aligned}\left(3x^7\right)\left(11x^6\right) &= 33x^{7+6}\\ &= 33x^{13}\end{aligned}$$

(b)
$$\begin{aligned}\frac{72y^8}{3y^9} &= 24y^{8-9}\\ &= 24y^{-1}\\ &= 24\left(\frac{1}{y}\right)\\ &= \frac{24}{y}\end{aligned}$$

(c) $$\left(2z^{-2}\right)^{-3}\left(z^{-6}\right)=\left(2\right)^{-3}\left(z^{-2}\right)^{-3}\left(z^{-6}\right)$$
$$=\frac{1}{2^3}\left(z^{(-2)\cdot(-3)}\right)\left(z^{-6}\right)$$
$$=\frac{1}{8}\left(z^{6}\right)\left(z^{-6}\right)$$
$$=\frac{1}{8}z^{6-6}$$
$$=\frac{1}{8}z^{0}$$
$$=\frac{1}{8}$$

Example 1.2 Simplify the following, writing your answers using positive exponents only.

(a) $\left(\dfrac{25p^6q^{-2}}{5p^8q^2}\right)^{-2}$ (b) $\dfrac{s^6}{3t^{-2}}\div\dfrac{-s^2}{33\left(-t\right)^2}$ (c) $\left(u^{-1}+v\right)^{-1}$

Solution

(a) $$\left(\frac{25p^6q^{-2}}{5p^8q^2}\right)^{-2}=\left(5p^{-2}q^{-4}\right)^{-2}$$
$$=5^{-2}p^4q^8$$
$$=\frac{1}{25}p^4q^8$$

(b) $$\frac{s^6}{3t^{-2}}\div\frac{-s^2}{33\left(-t\right)^2}=\frac{s^6}{3t^{-2}}\cdot\frac{33\left(-t\right)^2}{-s^2}$$
$$=\frac{s^6}{3t^{-2}}\cdot\frac{33t^2}{-s^2}$$ [Note that $\left(-t\right)^2\neq-t^2$]
$$=\frac{33s^6t^2}{-3s^2t^{-2}}$$
$$=-11s^4t^4$$

(c) $$\left(u^{-1}+v\right)^{-1}=\left(\frac{1}{u}+v\right)^{-1}$$
$$=\left(\frac{1+uv}{u}\right)^{-1}$$
$$=\frac{1}{\left(\frac{1+uv}{u}\right)}$$
$$=\frac{u}{1+uv}$$

4. Polynomials

Recall that an algebraic expression is formed by combining operations on variables. An important class of algebraic expressions are polynomials and the simplest polynomials are those that involve only one variable.

4.1 Polynomials in one variable

A polynomial in x of degree n is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n$ are real numbers with $a_n \neq 0$.

The expressions $a_k x^k$ are called the terms of the polynomial. The number a_0 is also more commonly known as the constant term. The real numbers $a_0, a_1, a_2, \ldots, a_n$ are called the coefficients of $1, x, x^2, \ldots, x^n$ respectively.

For example, consider the polynomial $2x^6 + 5x^4 + x^3 - 11x + 9$.

The degree of the polynomial is 6,
the terms of the polynomial are $2x^6$, $5x^4$, x^3, $-11x$ and 9,
the constant term is 9 and
the coefficients of x^6, x^4, x^3, x and 1 are 2,5,1, -11 and 9 respectively.

4.2 Operations on polynomials

Terms that have the same variable and exponent are called like terms, e.g. $7x^2$ and $10x^2$. Like terms may be combined by adding or subtracting their coefficients. For example, adding the like terms $8x$ and $3x$ will give $(8 + 3)x$ or $11x$. A polynomial expression is considered simplified when none of its terms are like terms.

Example 1.3 Simplify the following polynomial expressions.

(a) $\left(5x^4 + x^2 - 11x + 9\right) + \left(2x^4 - x^3 + 5x - 7\right)$

(b) $\left(5x^4 + x^2 - 11x + 9\right) - \left(2x^4 - x^3 + 5x - 7\right)$

Solution

(a) $\left(5x^4 + x^2 - 11x + 9\right) + \left(2x^4 - x^3 + 5x - 7\right)$

$= 5x^4 + x^2 - 11x + 9 + 2x^4 - x^3 + 5x - 7$

$= 5x^4 + 2x^4 - x^3 + x^2 - 11x + 5x + 9 - 7$ [group like terms together]

$= 7x^4 - x^3 + x^2 - 6x + 2$ [combine like terms]

(b) $\left(5x^4+x^2-11x+9\right)-\left(2x^4-x^3+5x-7\right)$

$= 5x^4+x^2-11x+9-2x^4+x^3-5x+7$ [note the sign changes]

$= 5x^4+2x^4-x^3+x^2-11x+5x+9-7$ [group like terms together]

$= 7x^4-x^3+x^2-6x+2$ [combine like terms]

Two polynomials may be multiplied together by applying the distributive law. Each term of one polynomial is multiplied by each term of the other. The resulting polynomial is then simplified by combining like terms.

Example 1.4 Simplify the following polynomial expressions.

(a) $x^2\left(3x^5-x+4\right)$

(b) $\left(x^2+5x\right)\left(2x^2-x-3\right)$

Solution

(a) $x^2\left(3x^5-x+4\right)=\left(x^2\right)\left(3x^5\right)+\left(x^2\right)\left(-x\right)+\left(x^2\right)\left(4\right)=3x^7-x^3+4x^2$

(b) $\left(x^2+5x\right)\left(2x^2-x-3\right)$

$=\left(x^2\right)\left(2x^2\right)+\left(x^2\right)\left(-x\right)+\left(x^2\right)\left(-3\right)+\left(5x\right)\left(2x^2\right)+\left(5x\right)\left(-x\right)+\left(5x\right)\left(-3\right)$

$=2x^4-x^3-3x^2+10x^3-5x^2-15x$

$=2x^4+9x^3-8x^2-15x$

In particular, there are some products of polynomials that are frequently used. Since these types of products are relatively common, it might be useful to memorise the formulas. The formulae are shown below.

$$\left(a+b\right)^2=a^2+2ab+b^2$$

$$\left(a-b\right)^2=a^2-2ab+b^2$$

$$\left(a+b\right)\left(a-b\right)=a^2-b^2$$

Example 1.5 Simplify the following polynomial expressions using the special product formulae.

(a) $\left(2x-3\right)^2$

(b) $\left(5y+1\right)^2-\left(2-5y\right)^2$

Solution

(a) $\left(2x-3\right)^2=\left(2x\right)^2-2\left(2x\right)\left(3\right)+3^2=4x^2-12x+9$

(b) $\left(5y+1\right)^2-\left(2-5y\right)^2 =\left[\left(5y+1\right)+\left(2-5y\right)\right]\left[\left(5y+1\right)-\left(2-5y\right)\right]$

$$
\begin{aligned}
&= (5y+1+2-5y)(5y+1-2+5y) \\
&= (3)(10y-1) \\
&= 30y-3
\end{aligned}
$$

4.3 Factoring polynomials

Factoring a polynomial is the process of expressing a polynomial as the product of two or more polynomials, i.e. we try to determine all the terms that were multiplied together to obtain the polynomial. For example,

$$5x^2-10x=5x(x-2)$$

The expressions $5x$ and $x-2$ are factors of $5x^2-10x$. A polynomial is considered to be completely factored if no further factors can be found.

The first step in factoring a polynomial will be factoring out its greatest common factor. This is achieved by observing all the terms to check if there is a factor that is common. If there is a common factor, we will factor it out. For example,

$$5x^2-10x=5x\cdot x-5x\cdot 2=5x(x-2)$$

Note that this is essentially a reverse application of the distributive law. After the greatest common factor has been factored out, we will need to use a factorisation method to factor the polynomial. Three methods are discussed below.

Method 1: Grouping

A polynomial can be factorised by rearranging and grouping terms so that a common term can be factored out.

Example 1.6 Factorise the following polynomials.

(a) $3ax+8y-3ay-8x$ (b) x^3+x^2+x+1

Solution

(a)
$$
\begin{aligned}
3ax+8y-3ay-8x &= 3ax-3ay-8x+8y && \text{[Rearrange the terms]} \\
&= 3a(x-y)-8(x-y) && \text{[Factor out common terms]} \\
&= (3a-8)(x-y)
\end{aligned}
$$

(b)
$$
\begin{aligned}
x^3+x^2+x+1 &= x^2(x+1)+x+1 \\
&= (x^2+1)(x+1)
\end{aligned}
$$

Method 2: Special formulae

There are some specific forms of polynomials that can be easily factorised using special formulae. It would be useful to memorise these specific forms.

$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$
$$a^2-b^2=(a+b)(a-b)$$
$$a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$$
$$a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

Example 1.7 Factorise the following polynomials.

(a) $25x^4-36x^2$ (b) $8x^3+125$

Solution

(a)
$$\begin{aligned}25x^4-36x^2&=x^2\left(25x^2-36\right)\\&=x^2\left[(5x)^2-6^2\right]\\&=x^2(5x+6)(5x-6)\end{aligned}$$

(b)
$$\begin{aligned}8x^3+125&=(2x)^3+5^3\\&=(2x+5)\left[(2x)^2-(2x)(5)+5^2\right]\\&=(2x+5)\left(4x^2-10x+25\right)\end{aligned}$$

Method 3: Trial and error

A polynomial of degree 2 with integer coefficients can be represented by the expression ax^2+bx+c. Some of these polynomials can be factored using a trial and error method. These factors can be written in the form $(px+q)(rx+s)$, where p, q, r and s are integers. A procedure to factor these polynomials is as follows.

1. Write all pairs of factors for a, the coefficient of x^2. One of these pairs will be the values of p and r.
2. Write all pairs of factors for c, the constant term. One of these pairs will be the values of q and s.
3. Try various combinations of these factors until the sum of the outer and inner terms, i.e. $psx+qrx$, is bx.
4. Check your answer by multiplying the factors.

Example 1.8 Factorise the following polynomials.

(a) $x^2 - 6x - 16$ (b) $3x^2 + 17x + 10$

Solution

(a) Since the coefficient of x^2 is 1, the only possible values of p and r are 1. The constant term is –16 and the possible pairs of q and s are (1)(–16), (–1)(16), (2)(–8), (–2)(8) and (4)(–4). The following table gives a list of the possible factors.

Possible factors	Sum of outer and inner terms
$(x+1)(x-16)$	$-15x$
$(x-1)(x+16)$	$15x$
$(x+2)(x-8)$	$-6x$
$(x-2)(x+8)$	$6x$
$(x+4)(x-4)$	0

Since bx is $-6x$, the factors of $x^2 - 6x - 16$ are $(x + 2)(x - 8)$.

(b) The coefficient of x^2 is 3 and thus possible values of p and r are 1 and 3. The constant term 10 has both positive and negative factors, but as the value of b is 17, which is positive, the values of q and s must be both positive. Positive factors are (1)(10) and (2)(5). The following table gives a list of the possible factors.

Possible factors	Sum of outer and inner terms
$(3x+1)(x+10)$	$31x$
$(3x+10)(x+1)$	$13x$
$(3x+2)(x+5)$	$17x$
$(3x+5)(x+2)$	$11x$

Since bx is $17x$, then $3x^2 + 17x + 10$ are $(3x + 2)(x + 5)$.

5 Rational expressions

So far, we have discussed addition, subtraction and multiplication of polynomials. What about the operation of division? We represent quotients of polynomials as a fraction in which the numerator and denominator are both polynomials. The resulting expressions are called rational expressions. Examples of rational expressions are

$$\frac{5x-1}{2x^2+3} \text{ and } \frac{5x^3-2x+11}{x-2}.$$

Because division by zero is undefined, the denominator of a rational expression must not be zero. Thus, in the second example above, x cannot take the value of 2.

5.1 Operations on rational expressions

Since rational expressions are essentially quotients, operations with rational expressions are conducted in the same way as quotients with real numbers. The four operations of addition, subtraction, multiplication and division are illustrated in Table 1.5.

Table 1.5: Rules of operation for rational expressions

No.	Rule	Example
1.	$\frac{P}{Q}\cdot\frac{R}{S}=\frac{PR}{QS}$	$\frac{x}{x+1}\cdot\frac{3x}{x-1}=\frac{(x)(3x)}{(x+1)(x-1)}=\frac{3x^2}{x^2-1}$
2.	$\frac{P}{Q}\div\frac{R}{S}=\frac{P}{Q}\cdot\frac{S}{R}=\frac{PS}{QR}$	$\frac{x}{x+1}\div\frac{3x}{x-1}=\frac{x}{x+1}\cdot\frac{x-1}{3x}=\frac{(x)(x-1)}{(x+1)(3x)}=\frac{x-1}{3(x+1)}$
3.	$\frac{P}{R}+\frac{Q}{R}=\frac{P+Q}{R}$	$\frac{x}{x+1}+\frac{3x}{x+1}=\frac{x+3x}{x+1}=\frac{4x}{x+1}$
4.	$\frac{P}{R}-\frac{Q}{R}=\frac{P-Q}{R}$	$\frac{x}{x+1}-\frac{3x}{x+1}=\frac{x-3x}{x+1}=\frac{-2x}{x+1}$

A rational expression is simplified if its numerator and denominator have no common factors other than 1 and –1. The process of simplification usually involves factoring the polynomials and using the property $\frac{ca}{cb}=\frac{a}{b}$ to remove the common factors. The next two examples will demonstrate this process.

Example 1.9 Simplify the following.

(a) $\frac{2x-6}{x+1}\cdot\frac{x^2+2x+1}{x^2-9}$ (b) $\frac{x^2+2x-3}{x^2+4x+3}\div\frac{x^2+4x-5}{x^2-1}$

Solution

(a)
$$\frac{2x-6}{x+1}\cdot\frac{x^2+2x+1}{x^2-9}=\frac{2(x-3)}{x+1}\cdot\frac{(x+1)(x+1)}{(x+3)(x-3)} \qquad \text{[Factor the polynomials]}$$
$$=\frac{2(x-3)(x+1)(x+1)}{(x+1)(x+3)(x-3)}$$
$$=\frac{2(x+1)}{x+3} \qquad \text{[Remove the common factors]}$$
$$=\frac{2x+2}{x+3}$$

(b) $$\frac{x^2+2x-3}{x^2+4x+3} \div \frac{x^2+4x-5}{x^2-1} = \frac{x^2+2x-3}{x^2+4x+3} \cdot \frac{x^2-1}{x^2+4x-5}$$
$$= \frac{(x+3)(x-1)}{(x+3)(x+1)} \cdot \frac{(x+1)(x-1)}{(x+5)(x-1)}$$
$$= \frac{(x-1)}{(x+1)} \cdot \frac{(x+1)}{(x+5)}$$
$$= \frac{(x-1)(x+1)}{(x+1)(x+5)}$$
$$= \frac{x-1}{x+5}$$

Example 1.10 Simplify the following.

(a) $\dfrac{1}{x}+\dfrac{1}{x-3}$ (b) $\dfrac{x}{x^2+7x+10}-\dfrac{3}{x^2-4}$

Solution

(a) $$\frac{1}{x}+\frac{1}{x-3} = \frac{x-3}{x(x-3)}+\frac{x}{x(x-3)}$$
$$= \frac{x-3+x}{x(x-3)}$$
$$= \frac{2x-3}{x(x-3)}$$

(b) $$\frac{x}{x^2+7x+10}-\frac{3}{x^2-4} = \frac{x}{(x+2)(x+5)}-\frac{3}{(x+2)(x-2)}$$
$$= \frac{x(x-2)}{(x+2)(x+5)(x-2)}-\frac{3(x+5)}{(x+2)(x-2)(x+5)}$$
$$= \frac{x(x-2)-3(x+5)}{(x+2)(x+5)(x-2)}$$
$$= \frac{x^2-2x-3x-15}{(x+2)(x+5)(x-2)}$$
$$= \frac{x^2-5x-15}{(x+2)(x+5)(x-2)}$$

5.2 Complex fractions

Complex fractions are rational expressions that contain fractions in its numerators and/or its denominators. Using the above techniques, the fractions in the numerators and/or denominators are usually simplified before simplifying the entire fraction.

Example 1.11 Simplify the expression $\dfrac{x-\dfrac{1}{x}}{x+\dfrac{1}{x}}$.

Solution

$$\begin{aligned}
\frac{x-\dfrac{1}{x}}{x+\dfrac{1}{x}} &= \frac{\dfrac{x^2}{x}-\dfrac{1}{x}}{\dfrac{x^2}{x}+\dfrac{1}{x}} \\
&= \frac{\dfrac{x^2-1}{x}}{\dfrac{x^2+1}{x}} \\
&= \frac{x^2-1}{x} \div \frac{x^2+1}{x} \\
&= \frac{x^2-1}{x} \times \frac{x}{x^2+1} \\
&= \frac{x^2-1}{x^2+1}
\end{aligned}$$

6. Rational exponents and radicals

Up to this point, we have only discussed expressions of the form a^n, where a is a real number and n is an integer. We will now explore expressions where the exponent n is a rational number.

6.1 n^{th} roots of real numbers

If n is a natural number and a is a real number, then the exponent expression

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

We read $a^{\frac{1}{n}}$ or $\sqrt[n]{a}$ as the n^{th} root of a. The notation $\sqrt[n]{a}$ is called a radical. In this section, we are particularly interested in radicals when $n = 2$ and $n = 3$. These roots are commonly referred to as square roots and cube roots respectively. For example,

$$4^{\frac{1}{2}} = \sqrt{4} = 2 \quad \text{and} \quad 8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$

Note that although 2^2 and $(-2)^2$ are both equal to 4, it would be a mistake to write $\sqrt{4}$ as ±2. The radical $\sqrt{4}$ denotes the principal square root of 4, which refers to the positive root 2 only.

6.2 Operations on radicals

As we can express radicals as rational exponents, we can derive the properties of radicals from the properties of exponents listed in Table 1.4. These properties are displayed in Table 1.6.

Table 1.6: Rules of operation for radicals

No.	Rule	Example
1.	$\sqrt[n]{a^n} = a$	$\sqrt{5^2} = 5$
2.	$\sqrt[n]{a^m} = \sqrt[n]{\left(a^m\right)} = \left(\sqrt[n]{a}\right)^m$	$\sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$
3.	$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$	$\sqrt{125} = \sqrt{(25)(5)} = \sqrt{25} \cdot \sqrt{5} = 5\sqrt{5}$
4.	$\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt{\dfrac{25}{36}} = \dfrac{\sqrt{25}}{\sqrt{36}} = \dfrac{5}{6}$

A frequent error often committed concerns the addition and subtraction of radicals. The radical operation $\sqrt[n]{a \pm b}$ is not equal to $\sqrt[n]{a} \pm \sqrt[n]{b}$. For example, consider $\sqrt{1+1} = \sqrt{2}$. This is not equal to $\sqrt{1} + \sqrt{1} = 2$. As radicals can be expressed as exponent expressions, this is equivalent to stating that generally,

$$(a+b)^{\frac{1}{n}} \text{ is } \textbf{not} \text{ equal to } a^{\frac{1}{n}} + b^{\frac{1}{n}}.$$

When working with radicals, it is normally easier to do operations when we first simplify the radical. A radical $\sqrt[n]{a}$ is simplified when

(a) the powers of all factors of a are less than n and
(b) no radical appears in a denominator.

The examples below will clarify the definition.

Example 1.12 Simplify the following radicals.

(a) $\sqrt{300}$ (b) $\sqrt[3]{8x^4y^5}$

Solution

(a) $\sqrt{300} = \sqrt{100 \cdot 3}$

$= \sqrt{100} \cdot \sqrt{3}$ [$100 = 10^2$]

$= 10\sqrt{3}$ [The powers of all factors are less than 2]

(b) $$\begin{aligned}\sqrt[3]{8x^4y^5} &= \sqrt[3]{8}\cdot\sqrt[3]{x^4y^5}\\ &= 2\sqrt[3]{\left(x^3y^3\right)xy^2}\\ &= 2\sqrt[3]{(xy)^3\,xy^2}\\ &= 2xy\sqrt[3]{xy^2} \qquad \text{[The powers of all factors are less than 3]}\end{aligned}$$

When a radical appears in a denominator, we can use a technique called rationalizing the denominator to eliminate it. The technique involves both the radical property $\sqrt[n]{a^n} = a$ and/or the algebraic formula $(x + y)(x - y) = x^2 - y^2$. In general,

1. If the denominator is of the form $\sqrt{a}$, we multiply the radical by $\dfrac{\sqrt{a}}{\sqrt{a}}$.
2. If the denominator is of the form $a+\sqrt{b}$, we multiply the radical by $\dfrac{a-\sqrt{b}}{a-\sqrt{b}}$.
3. If the denominator is of the form $a-\sqrt{b}$, we multiply the radical by $\dfrac{a+\sqrt{b}}{a+\sqrt{b}}$.

Example 1.13 Simplify the following radicals.

(a) $\dfrac{1}{\sqrt{x}}$ (b) $\dfrac{3}{5-2\sqrt{y}}$

Solution

(a) $$\begin{aligned}\frac{1}{\sqrt{x}} &= \frac{1}{\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}}\\ &= \frac{\sqrt{x}}{\sqrt{x^2}}\\ &= \frac{\sqrt{x}}{x} \qquad \text{[There are no radicals in the denominator]}\end{aligned}$$

(b) $$\begin{aligned}\frac{3}{5-2\sqrt{y}} &= \frac{3}{5-2\sqrt{y}}\cdot\frac{5+2\sqrt{y}}{5+2\sqrt{y}}\\ &= \frac{3\left(5+2\sqrt{y}\right)}{5^2-\left(2\sqrt{y}\right)^2}\\ &= \frac{15+6\sqrt{y}}{25-4y} \qquad \text{[There are no radicals in the denominator]}\end{aligned}$$

Example 1.14 Simplify $\dfrac{1}{\sqrt{x}+\sqrt{y}}+\dfrac{1}{\sqrt{x}-\sqrt{y}}$.

Solution

To simplify addition and subtraction of rational expressions that include radicals, we can change the denominators to a common expression first, followed by rationalizing the denominator, if necessary.

$$\begin{aligned}\frac{1}{\sqrt{x}+\sqrt{y}}+\frac{1}{\sqrt{x}-\sqrt{y}} &= \frac{\left(\sqrt{x}-\sqrt{y}\right)}{\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)}+\frac{\left(\sqrt{x}+\sqrt{y}\right)}{\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)} \\ &= \frac{\sqrt{x}-\sqrt{y}}{\left(\sqrt{x}\right)^2-\left(\sqrt{y}\right)^2}+\frac{\sqrt{x}+\sqrt{y}}{\left(\sqrt{x}\right)^2-\left(\sqrt{y}\right)^2} \\ &= \frac{\sqrt{x}-\sqrt{y}+\sqrt{x}+\sqrt{y}}{x-y} \\ &= \frac{2\sqrt{x}}{x-y}\end{aligned}$$

7. Discussion questions

1. Simplify the following expressions. Give your answer in positive exponents only.

(a) $\left(5a^{-2}\right)\left(5a\right)^{-2}$

(b) $\left(4ab^{2}\right)^{2}\left(a^{-1}b^{-1}\right)^{3}$

(c) $\dfrac{25p^{3}q^{2}}{5p^{2}q^{-1}}\cdot p^{5}q^{5}$

(d) $\dfrac{5u^{8}v^{-2}}{u^{-1}v^{6}}\div\left(5u^{2}\right)^{0}$

2. Simplify the following expressions.

(a) $\left(x^{2}+1\right)\left(3x^{3}-2\right)$

(b) $\left(7x^{3}-x\right)^{2}$

(c) $\left(2x+5\right)^{2}-\left(2x-1\right)^{2}$

(d) $x^{2}\left(x-1\right)\left(x+1\right)-\left(x^{2}+2\right)^{2}$

3. Factorise the following expressions.

(a) $5ax-12ay+10x-24y$

(b) $50x^{3}-98x$

(c) $64y^{3}-1$

(d) $2x^{2}-3x-2$

4. Simplify the following expressions.

(a) $\dfrac{3x+12}{4x-2}\cdot\dfrac{8x-4}{3}$

(b) $\dfrac{x-1}{x+3}\div\dfrac{x^{2}-2x+1}{x^{2}-9}$

(c) $\dfrac{7}{x^{2}-8x+12}-\dfrac{1}{x^{2}-36}$

(d) $\dfrac{2-\dfrac{6}{x+5}}{1-\dfrac{x}{x+2}}$

5. Simplify the following radicals.

(a) $\sqrt{25u^{6}v^{4}}\cdot\sqrt{4w^{10}}$

(b) $\dfrac{2}{\sqrt[3]{x}}$

(c) $\dfrac{8}{3-2\sqrt{x}}$

(d) $\sqrt{x+1}-\dfrac{1}{\sqrt{x+1}}$

8. Supplementary questions

1. Simplify the following expressions. Give your answer in positive exponents only.

(a) $\left(3a^{-2}b^{8}\right)^{2}$ (b) $\left(2p^{-4}\right)^{2}\left(4q^{-1}\right)^{3}\left(2r\right)^{-2}$

(c) $\dfrac{5^{2}u^{-3}\left(v^{-3}\right)^{4}}{8^{2}\left(u^{-5}v^{4}\right)^{3}}$ (d) $\left(\dfrac{8x^{6}y^{6}}{5x^{4}y^{3}}\right)^{-1}\cdot\left(125xy\right)^{0}$

2. Simplify the following expressions.

(a) $6a\left(2a-3\right)+a\left(a-1\right)$ (b) $\left(0.6p+1.2q\right)\left(0.4p-2.1q\right)$

(c) $\left(4u-3v\right)\left(4u+3v\right)+\left(u-v\right)^{2}$ (d) $x\left\{5\left[x-\left(5-x\right)+\left(x+1\right)\left(x-5\right)\right]\right\}$

3. Factorise the following expressions.

(a) $2ax+8ay+bx+4by$ (b) $16u^{4}-81v^{4}$

(c) $y^{2}+2y-80$ (d) $5z^{2}-14z-3$

4. Simplify the following expressions.

(a) $\dfrac{9r^{2}-12r-5}{3r+6}\cdot\dfrac{6r+12}{6r+2}$ (b) $\dfrac{u^{2}-3u-18}{u^{2}-4u-12}\div\dfrac{u^{2}-9u+14}{u^{2}-5u-14}$

(c) $\dfrac{3}{x^{2}-8x+16}-\dfrac{2}{x^{2}-16}$ (d) $\dfrac{x}{5-x}+\dfrac{4x+3}{x^{2}-25}$

5. Simplify the following radicals.

(a) $\sqrt[3]{m^{15}n^{3}p^{18}}$ (b) $\dfrac{6}{\sqrt{(x+1)}}$

(c) $\dfrac{3}{\sqrt{x}-2}$ (d) $\left(x-1\right)^{\frac{1}{2}}+\dfrac{1}{5}x\left(x-1\right)^{-\frac{1}{2}}$

BUSINESS MATHEMATICS

SESSION 2

FUNDAMENTALS OF ALGEBRA II

At the end of the session, students should be able to:

1. Solve linear and quadratic equations.
2. Solve linear, quadratic and absolute value inequalities.
3. Find terms and compute partial sums of an arithmetic or geometric progression.

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1. Introduction

This session continues on the previous chapter by reviewing further algebra concepts that you will use in this course. It covers methods to solve linear, rational-linear, quadratic equations and linear, quadratic, rational and absolute value inequalities. In addition, the concepts of arithmetic and geometric progressions are discussed.

Many real life situations can be expressed as equations or inequalities. For example, a salesperson monthly salary that is based on a basic salary and sales commission can be written as a linear equation. Or the height of a ball being thrown upwards can be written as a quadratic equation. At other times, we may be interested in purchasing a stock when the price falls below a certain value. Being able to solve these equations and inequalities algebraically will be very useful in your work.

Similarly, arithmetic and geometric progressions are specific type of sequences of numbers which used frequently to model real-life phenomena. For example, a manufacturing company might decide to increase its output by a certain quantity for the next few years. Or the inflation rate of currencies could follow a geometric pattern. Whatever the situation, it will be useful to be able to recognize and perform operations on these sequences.

2. Solving equations

Recall that an algebraic expression is formed by combining operations on variables. An equation is a mathematical statement that indicates that two algebraic expressions are equal. For example, if we state that the expression $2x - 3$ is exactly the same as $x + 1$, then we can write this as

$$2x - 3 = x + 1$$

The solution to an equation is a number or numbers that, when replaced with the variable, makes the mathematical equation a true statement. In the above example, the number 4 is a solution of the equation because if the variable x is replaced by the number 4, the statement is true, i.e. $2(4) - 3 = 4 + 1$.

2.1 Linear equations

A linear (or first degree) equation in one variable is an equation in which the highest power of the variable is 1. It can be written in the form

$$ax + b = 0.$$

To solve a linear equation, we will have to isolate the variable. This means getting the variable by itself on one side of the equal sign. To do so, we will use the equality property of real numbers.

Example 2.1 Solve the linear equation $7x - 3 = 2(x + 1)$.

Solution

$$\begin{aligned}
7x-3 &= 2(x+1) \\
7x-3 &= 2x+2 && \text{[Simplify the expressions on both sides]} \\
7x-3-2x &= 2x+2-2 && \text{[Subtract } 2x \text{ from both sides]} \\
5x-3 &= 2 \\
5x-3+3 &= 2+3 && \text{[Add 3 to both sides]} \\
5x &= 5 \\
\frac{1}{5}(5x) &= \frac{1}{5}(5) && \text{[Divide both sides by 5]} \\
x &= 1
\end{aligned}$$

Sometimes a linear equation contains fractions. To solve these types of equations, we usually begin by multiplying each term of the equation by the lowest common denominator.

Example 2.2 Solve the equation $\frac{2x}{3} - \frac{x}{4} = 10$.

Solution

$$\begin{aligned}
\frac{2x}{3}-\frac{x}{4} &= 10 && \text{[The lowest common denominator of 3 and 4 is 12]} \\
12\left(\frac{2x}{3}-\frac{x}{4}\right) &= 12(10) && \text{[Multiply both sides by 12]} \\
12\left(\frac{2x}{3}\right)-12\left(\frac{x}{4}\right) &= 120 \\
8x-3x &= 120 \\
5x &= 120 \\
x &= 24
\end{aligned}$$

Example 2.3 Solve the equation $\frac{2}{3x-1}=\frac{1}{x+1}$.

Solution

$$\frac{2}{3x-1}=\frac{1}{x+1} \qquad \text{[The lowest common denominator is } (3x-1)(x+1)\text{]}$$

$$(3x-1)(x+1)\frac{2}{3x-1}=(3x-1)(x+1)\frac{1}{x+1}$$

$$2(x+1)=1(3x-1)$$

$$2x+2=3x-1$$

$$2x-3x=-1-2$$

$$-x=-3 \qquad \text{[Mulitply both sides by } -1\text{]}$$

$$x=3$$

2.2 Quadratic equations

A quadratic (or second degree) equation in one variable is an equation in which the highest power of the variable is 2. It can be written in the form

$$ax^2+bx+c=0.$$

To solve a quadratic equation, we can use factorization, completing the square or the quadratic formula.

Method 1: Factorization

A procedure to solve a quadratic equation by factorization is as follows.

1. Ensure that the right side of the equation is 0.
2. Factorize the quadratic expression using trial and error.
3. Use the property that if $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

Example 2.4 Solve the quadratic equation $x^2 + 2 = 3x$.

Solution

$$x^2+2=3x \qquad \text{[Highest power is 2 so this is a quadratic equation]}$$

$$x^2+2-3x=3x-3x$$

$$x^2-3x+2=0 \qquad \text{[Ensure the right side of the equation is 0]}$$

$$(x-1)(x-2)=0 \qquad \text{[Factorize the expression using trial and error]}$$

$$x-1=0 \quad or \quad x-2=0$$

$$x=1 \quad or \quad x=2$$

Method 2: Completing the square

A procedure to solve a quadratic equation by completing the square is as follows.

1. Rearrange the equation $ax^2 + bx + c = 0$ in the form $x^2 + \frac{b}{a}x = -\frac{c}{a}$.
2. Square half the coefficient of x and add it to both sides, i.e.
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$
3. Factorise the left side of the equation. Square root both sides and solve for x.

Example 2.5 Solve the quadratic equation $2x^2 + 5x - 12 = 0$ by completing the square.

Solution

$$2x^2 + 5x - 12 = 0$$

$$x^2 + \frac{5}{2}x - 6 = 0$$

$$x^2 + \frac{5}{2}x = 6$$ [Rewrite the equation into the form $x^2 + \frac{b}{a}x = -\frac{c}{a}$]

$$x^2 + \frac{5}{2}x + \left(\frac{5}{2(2)}\right)^2 = 6 + \left(\frac{5}{2(2)}\right)^2$$ [Square half the coefficient of x and add it to both sides]

$$x^2 + \frac{5}{2}x + \frac{25}{16} = \frac{121}{16}$$

$$\left(x + \frac{5}{4}\right)^2 = \frac{121}{16}$$ [Factorise the left side of the equation]

$$x + \frac{5}{4} = \frac{11}{4} \quad or \quad -\frac{11}{4}$$ [Square root both sides and solve the equation]

$$x = \frac{11}{4} - \frac{5}{4} \quad or \quad -\frac{11}{4} - \frac{5}{4}$$

$$= \frac{3}{2} \quad or \quad -4$$

Method 3: Quadratic formula

When we apply completing the square to solve the quadratic equation $ax^2 + bx + c = 0$, we will obtain a general formula that can be used to solve any quadratic equation. This formula is known as the quadratic formula and the solutions for the equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 2.6 Solve the quadratic equation $0.01x^2 + 0.15x - 0.3 = 0$.

Solution
Comparing the equation with the general form, $a = 0.01$, $b = 0.15$, $c = -0.3$.

$$\begin{aligned} x &= \frac{-0.15 \pm \sqrt{(0.15)^2 - 4(0.01)(-0.3)}}{2(0.01)} \\ &= \frac{-0.15 \pm \sqrt{0.0345}}{0.02} \\ &= 1.79 \quad or \quad -16.79 \end{aligned}$$

3. Solving inequalities

An inequality is a mathematical statement that indicates a relation between two expressions. The relation can be expressed in terms of "greater than" or "less than". For example, 2 is greater than 1, so we write $2 > 1$. The number -3 is less than -1, so we write $-3 < -1$. When we substitute algebraic expressions for numbers, we can also represent relations between expressions. For example, we can state that the expression $2x - 1$ is less than 7 as $2x - 1 < 7$.

The solution to an inequality is a set of numbers such that, when any number in the set is replaced with the variable, the inequality becomes true. For example, given the inequality $2x - 1 < 7$, the solution would be the set of real numbers that are less than 4. If the variable x is replaced by any number which is less than 4, the statement is true. If x is replaced by a number which is greater than 4, then the statement is not true. We would need a notation to represent these set of numbers.

3.1 Intervals

Sets of real numbers are usually denoted by finite or infinite intervals. Finite intervals consist of all numbers that lie between two fixed real numbers, which constitute the end points of the interval. A closed interval $[a, b]$ will include its endpoints whereas an open interval (a, b), does not include its endpoints. A half open interval, $[a, b)$ or $(a, b]$, will include only one end point. These different types with their inequality equivalent are illustrated in Table 2.1 below.

Table 2.1: Finite intervals

Interval	Inequality	Example (Interval)	Example (Inequality)
Closed: $[a,b]$	$a \le x \le b$	$[1,4]$	$1 \le x \le 4$
Open: (a,b)	$a < x < b$	$(1,4)$	$1 < x < 4$
Half-open: $[a,b)$	$a \le x < b$	$[1,4)$	$1 \le x < 4$
Half-open: $(a,b]$	$a < x \le b$	$(1,4]$	$1 < x \le 4$

Infinite intervals are intervals of infinite length. They consist of the notation ∞, called infinity. Note that infinity is not a number, it is used to represent that the interval has no limit. Hence they are always written with an open interval. Hence the open interval (–∞, ∞) would be the set of all real numbers. The different types of infinite intervals with their inequality equivalent are illustrated in Table 2.2 below.

Table 2.2: Infinite intervals

Interval	Inequality	Example (Interval)	Example (Inequality)
$[a,\infty)$	$x \geq a$	$[3,\infty)$	$x \geq 3$
(a,∞)	$x > a$	$(3,\infty)$	$x > 3$
$(-\infty,a]$	$x \leq a$	$(-\infty,3]$	$x \leq 3$
$(-\infty,a)$	$x < a$	$(-\infty,3)$	$x < 3$

3.2 Linear inequalities

A linear inequality in one variable is an inequality in which the highest power of the variable is 1. Solving a linear inequality is similar to solving a linear equation, i.e. we isolate the variable. In order to isolate the variable, we will use the properties listed in Table 2.3 below.

Table 2.3: Properties of inequalities

No.	Property	Example
1.	If $a<b$ and $b<c$, then $a<c$.	$1<2$ and $2<3$, so $1<3$.
2.	If $a<b$, then $a+c<b+c$.	$x+1<5$, so $x+1-1<5-1$ that is $x<4$
3.	If $a<b$ and $c>0$, then $ac<bc$.	$1<2$, so $1(3)<2(3)$ that is $3<6$
4.	If $a<b$ and $c<0$, then $ac>bc$.	$1<2$, so $1(-3)>2(-3)$ that is $-3>-6$

Example 2.7 Solve the linear inequality –5x – 1 > 9.

Solution

$$-5x-1>9$$

$$-5x-1+1>9+1 \qquad \text{[Add 1 to both sides]}$$

$$-5x>10$$

$$-\frac{1}{5}(-5x)<-\frac{1}{5}(10) \qquad \text{[Divide by } -5 \text{ on both sides. Note the change in inequality sign]}$$

$$x<-2$$

The solution set would be all x values in the interval (–∞, 2).

Example 2.8 Solve the linear inequality $1 < 2x - 3 \leq 13$.

Solution

$$1 < 2x - 3 \leq 13$$
$$4 < 2x \leq 16 \qquad \text{[Add 3 to all parts of the inequality]}$$
$$2 < x \leq 8 \qquad \text{[Divide by 2 on all parts of the inequality]}$$

The solution set would be all x values in the interval (2, 8].

3.3 Quadratic inequalities

A quadratic inequality in one variable is an inequality in which the highest power of the variable is 2. Solving a quadratic inequality involves two steps: 1) Factorizing the quadratic expression and 2) Constructing a sign diagram to determine the intervals that satisfy the inequality.

Example 2.9 Solve the quadratic inequality $x^2 - 2x < 3$.

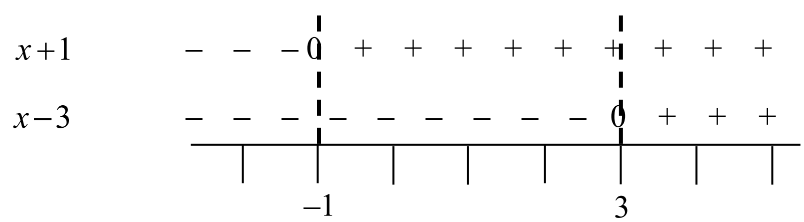
Solution

$$x^2 - 2x - 3 < 0 \qquad \text{[Set the right side of the inequality to be 0]}$$
$$(x-3)(x+1) < 0 \qquad \text{[Factorize the quadratic expression]}$$

Equating the two factors to zero, we obtain two critical values of x.

$$x - 3 = 0 \Rightarrow x = 3 \qquad \text{and} \qquad x + 1 = 0 \Rightarrow x = -1$$

Next, the two values are indicated on a number line. A sign diagram is then constructed to check the values of x that will cause the two factors to be positive or negative.



The sign diagram shows that when x is less than 3, $x - 3$ would be negative and when x is greater than 3, $x - 3$ would be positive. Similarly, if x is less than -1, $x + 1$ would be negative and if x is greater than -1, $x + 1$ would be positive.

We need values of x such that the product of the two factors are negative because that was the initial question, i.e. $(x - 3)(x + 1) < 0$. These values of x must cause the factors to have different signs, i.e. one is positive and the other is negative. Thus from the sign diagram, we see this occurs only when $-1 < x < 3$.

Thus the solution set would be all x values in the interval $(-1, 3)$.

Example 2.10 Solve the quadratic inequality $x^2 \geq 4$.

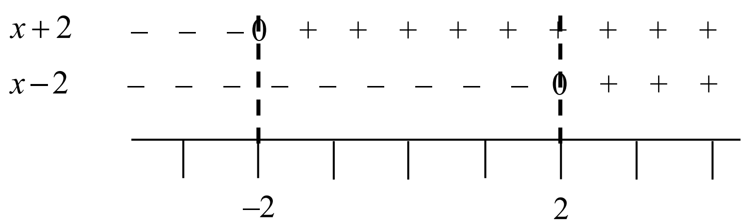
Solution

$$x^2 - 4 \geq 0$$
$$(x+2)(x-2) \geq 0$$

Equating the two factors to zero, we obtain two critical values of x.

$$x-2=0 \Rightarrow x=2 \qquad \text{and} \qquad x+2=0 \Rightarrow x=-2$$

Next, the two values are indicated on a number line. A sign diagram is then constructed.



The sign diagram shows that when x is less than 2, $x - 2$ would be negative and when x is greater than 2, $x - 2$ would be positive. Similarly, if x is less than –2, $x + 2$ would be negative and if x is greater than –2, $x + 2$ would be positive.

We need values of x such that the product of the two factors are positive or zero because that was the initial question, i.e. $(x - 2)(x + 2) \geq 0$. These values of x must cause the factors to have the same signs, i.e. both are positive or both are negative. Thus from the sign diagram, we see this occurs only when $x \leq -2$ or $x \geq 2$.

Thus the solution set would be all x values in the interval $(-\infty, -2] \cup [2, \infty)$.

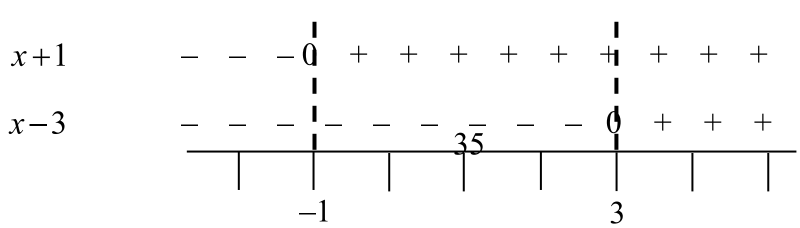
Sometimes an inequality contains a quotient of two linear algebraic expressions. The method to solve these types of inequalities is similar to the method used to solve quadratic inequalities. The main difference occurs when we recall that the denominator of a quotient cannot be equal to 0.

Example 2.11 Solve the inequality $\dfrac{x+1}{x-3} \leq 0$.

Solution
Equating the two factors to zero, we obtain two critical values of x.

$$x-3=0 \Rightarrow x=3 \qquad \text{and} \qquad x+1=0 \Rightarrow x=-1$$

We follow the same procedure of constructing a sign diagram as in Example 2.9.



Now, we need values of x such that the quotient of the two factors are negative or zero because that was the initial question, i.e. $\frac{x+1}{x-3} \leq 0$. From the sign diagram, these values of x fall between -1 and 3, i.e. $-1 \leq x \leq 3$.

However, take note that as the denominator of the quotient cannot be 0, the value of x cannot be equal to 3. So we modify the solution to $-1 \leq x < 3$.

Thus the solution set would be all x values in the interval $[-1, 3)$.

3.3 Absolute value inequalities

The absolute value of a number a is denoted by $|a|$ and is defined by

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}.$$

It is used to represent the magnitude of the number. Thus $|5|$ is equal to 5 and $|-5|$ is also equal to 5. Geometrically, $|a|$ is the distance between the origin and the point on the number line that represents a.

We can transform a negative number to its positive counterpart by using the definition provided. Hence we have $|-5| = -(-5) = 5$.

Example 2.12 Evaluate (a) $|\pi + 8| + 1$ and (b) $|\pi - 8| + 1$.

Solution

(a) Since $\pi + 8$ is positive, $|\pi + 8| = \pi + 8$.

Thus $|\pi + 8| + 1 = \pi + 8 + 1$
$= \pi + 9$

(b) Since $\pi - 8$ is negative, $|\pi - 8| = -(\pi - 8) = 8 - \pi$.

Thus $|\pi - 8| + 1 = 8 - \pi + 1$
$= 9 - \pi$

Some properties of absolute value are given in Table 2.4.

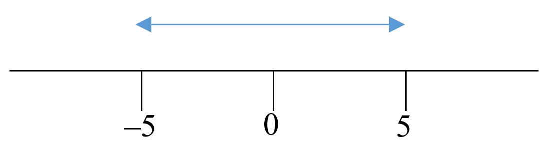
Table 2.4: Properties of absolute value

No.	Property	Example
1.	$\lvert -a \rvert = \lvert a \rvert$	$\lvert -2 \rvert = 2, \lvert 2 \rvert = 2$
2.	$\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	$\lvert (-2)(3) \rvert = \lvert -6 \rvert = 6, \lvert (-2) \rvert \lvert (3) \rvert = (2)(3) = 6$
3.	$\left\lvert \frac{a}{b} \right\rvert = \frac{\lvert a \rvert}{\lvert b \rvert}$	$\left\lvert \frac{-2}{3} \right\rvert = \left\lvert -\frac{2}{3} \right\rvert = \frac{2}{3}, \frac{\lvert -2 \rvert}{\lvert 3 \rvert} = \frac{2}{3}$
4.	$\lvert a+b \rvert \le \lvert a \rvert + \lvert b \rvert$ (Triangle inequality)	$\lvert 3+(-2) \rvert = \lvert 1 \rvert = 1 \le \lvert 3 \rvert + \lvert -2 \rvert = 3+2 = 5$

To solve absolute value inequalities, we first consider the cases when $|x| < k$ and $|x| > k$.

Example 2.13 Solve the inequalities (a) $|x| \le 5$ and (b) $|x| \ge 5$.

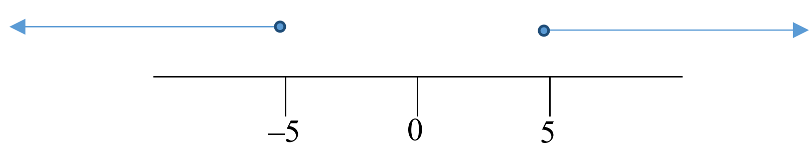
Solution
(a) $|x| \le 5$ implies that the distance from the origin to the number 5 on the number line is less than or equal to 5. This gives the following representation.



From the number line, we can see that values equal to and between –5 and 5 will satisfy the inequality, i.e. $|x| \le 5 \Rightarrow -5 \le x \le 5$.

Thus the solution set would be all x values in the interval [–5, 5].

(b) $|x| \ge 5$ implies that the distance from the origin to the number 5 on the number line is greater than or equal to 5. This gives the following representation.



From the number line, we can see that $|x| \ge 5 \Rightarrow x \ge 5$ or $x \le -5$.

Thus the solution set would be all x values in the interval $(-\infty, -5] \cup [5, \infty)$.

From Example 2.13, we can generalize absolute value inequalities to two formulae.

$$\lvert x \rvert \le k \Rightarrow -k \le x \le k \text{ and } \lvert x \rvert \ge k \Rightarrow x \le -k \quad or \quad x \ge k$$

Example 2.14 Solve the inequalities (a) $|2x - 1| \le 3$ and (b) $|2x - 7| \ge 9$.

Solution

(a) Since the inequality sign is ≤, we use $|x| \le k \Rightarrow -k \le x \le k$

$$\begin{aligned} |2x-1| \le 3 \Rightarrow -3 \le 2x-1 \le 3 \\ -2 \le 2x \le 4 \\ -1 \le x \le 2 \end{aligned}$$

Thus the solution set would be all x values in the interval $[-1, 2]$.

(b) Since the inequality sign is ≥, we use $|x| \ge k \Rightarrow x \ge k$ or $x \le -k$.

$$\begin{aligned} |2x-7| \ge 9 \Rightarrow 2x-7 \ge 9 \quad or \quad 2x-7 \le -9 \\ 2x \ge 16 \quad or \quad 2x \le -2 \\ x \ge 8 \quad or \quad x \le -1 \end{aligned}$$

Thus the solution set would be all x values in the interval $(-\infty, -1] \cup [8, \infty)$.

4. Arithmetic and geometric progressions

4.1 Arithmetic progression

An arithmetic progression is a sequence of numbers in which the subsequent number is formed by adding a constant to the preceding number. The n^{th} number in the sequence is known as a term of the sequence, denoted by a_n, and the constant is called the common difference, d.

For example, in the sequence below,

$$\begin{array}{ccccccccc} a_1 & & a_2 & & a_3 & & a_4 & & a_{10} \\ 2 & \xrightarrow{+3} & 5 & \xrightarrow{+3} & 8 & \xrightarrow{+3} & 11 & \ldots & 29 \end{array}$$

First term $a_1 = 2$, Second term $a_2 = 5$, Third term $a_3 = 8$, …, Tenth term $a_{10} = 29$
Common difference $d = 3$.

An arithmetic progression is completely defined by the first term a and the common difference d. At times, we may want to be able to know what is the value of a term down the sequence. Instead of working out the entire sequence, we can derive a formula for this. The formula is derived through spotting a recurrent pattern in the sequence.

Observe that if the first term is a and the common difference is d, then

$$
\begin{aligned}
a_1 &= a \\
a_2 &= a + d \\
a_3 &= a + d + d = a + 2d \\
a_4 &= a + d + d + d = a + 3d \\
a_5 &= a + 4d \\
&\vdots
\end{aligned}
$$

We can thus obtain the formula for the n^{th} term of an arithmetic sequence.

$$a_n = a + (n-1)d$$

We may also be interested in the sum of the first n terms of an arithmetic sequence. We use the letter S_n to denote the sum of a sequence. Thus we have

$$
\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \ldots + a_n
\end{aligned}
$$

The formula for the sum of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Example 2.15 In the arithmetic sequence, 4, 9, 14, 19, … find the 20^{th} term of the sequence and the sum of the first 20 terms.

Solution
In the sequence, $a = 4$, $d = 5$. Setting $n = 20$, we obtain the following.

$$
\begin{aligned}
a_{20} &= 4 + (20-1)(5) \\
&= 99
\end{aligned}
$$

$$
\begin{aligned}
S_{20} &= \frac{20}{2}\left[2(4) + (20-1)(5)\right] \\
&= 1{,}030
\end{aligned}
$$

4.2 Geometric progression

A geometric progression is a similar sequence of numbers in which the subsequent term is formed by multiplying a constant to the preceding term. The multiplicative constant is called the common ratio, *r*. For example, in the sequence below,

$$\begin{array}{ccccccccc} a_1 & & a_2 & & a_3 & & a_4 & & a_{10} \\ 2 & & 6 & & 18 & & 54 & \ldots & 39{,}366 \\ & \times 3 & & \times 3 & & \times 3 & & & \end{array}$$

First term $a_1 = 2$, Second term $a_2 = 6$, Third term $a_3 = 18$, …, Tenth term $a_{10} = 39{,}366$.
Common difference $r = 3$.

Similar to an arithmetic progression, a geometric progression is completely defined by the first term and the common ratio. Observe that if the first term is a and the common ratio is r, then

$$\begin{aligned} a_1 &= a \\ a_2 &= ar \\ a_3 &= ar \times r = ar^2 \\ a_4 &= ar^2 \times r = ar^3 \\ a_5 &= ar^4 \\ &\vdots \end{aligned}$$

We can thus obtain the formula for the n^{th} term of a geometric sequence.

$$a_n = ar^{n-1}$$

We may also be interested in the sum of the first n terms of a geometric sequence. Similar to the arithmetic sequence, we use the letter S_n to denote the sum of a sequence, i.e.

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \ldots + a_n \end{aligned}$$

The formula for the sum of the first n terms of a geometric sequence is given by

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r} \quad \text{where } r \neq 1$$

Example 2.16 In the geometric sequence, 3, 6, 12, 24, … find the 10th term of the sequence and the sum of the first 10 terms.

Solution
In the sequence, $a = 3$, $r = 2$. Setting $n = 10$, we obtain the following.

$$\begin{aligned} a_{10} &= 3(2)^{10-1} \\ &= 3(2)^9 \\ &= 1{,}536 \end{aligned}$$

$$\begin{aligned} S_{10} &= \frac{3\left(1-2^{10}\right)}{1-2} \\ &= 3{,}069 \end{aligned}$$

Example 2.17 A software company had sales of \$1 million in its first year of operation.

(a) If the sales increased by \$200,000 per year thereafter, find the company's sales in the fifth year and its total sales over the first 5 years of operation.

(b) If the sales increased by 10% per year thereafter, find the company's sales in the fifth year and its total sales over the first 5 years of operation.

Solution
(a) The company's sales follow an arithmetic progression with a = \$1,000,000, d = \$200,000. Setting $n = 5$,

$$\begin{aligned} \text{Sales in the fifth year, } a_5 &= 1{,}000{,}000 + (5-1)(200{,}000) \\ &= \$1{,}800{,}000 \end{aligned}$$

$$\begin{aligned} \text{Total sales over five years, } S_5 &= \frac{5}{2}\left[2(1{,}000{,}000) + (5-1)(200{,}000)\right] \\ &= \$7{,}000{,}000 \end{aligned}$$

(b) The company's sales follow a geometric progression with a = \$1,000,000, $r = 1.1$. Setting $n = 5$,

$$\begin{aligned} \text{Sales in the fifth year, } a_5 &= 1{,}000{,}000(1.1)^{5-1} \\ &= \$1{,}464{,}100 \end{aligned}$$

$$\begin{aligned} \text{Total sales over five years, } S_5 &= \frac{1{,}000{,}000\left(1-1.1^5\right)}{1-1.1} \\ &= \$6{,}105{,}100 \end{aligned}$$

5. Discussion questions

1. Solve the following equations.

(a) $7x-1=6x+13$

(b) $\frac{1}{2}x+\frac{4}{3}=\frac{2}{3}x-2$

(c) $\frac{5x-1}{3x-2}=3$

(d) $x^2-7x+10=0$

(e) $0.01x^2+0.2x-3.51=0$

(f) $\frac{2x+3}{x+1}=\frac{x-1}{x}$

2. A model rocket is launched vertically upward so that its height h, in meters, t seconds after launch is given by the equation $h = -5t^2 + 126t + 1$. Find the time(s) when the rocket is at a height of 400 meters.

3. Solve the following inequalities.

(a) $-4x \le 24$

(b) $0<\frac{1}{2}x-1<7$

(c) $x^2-2x-15\ge 0$

(d) $\frac{2x-1}{x+7}\le 0$

(e) $|5x-11|\le 14$

(f) $|3x+1|>17$

4. The relationship between the temperature in degree Celsius (ºC) and degree Fahrenheit (ºF) is given by the formula $C=\frac{5}{9}(F-32)$. If the range of temperature in Singapore is between 24 ºC and 35 ºC, find the range in ºF.

5(a) In the arithmetic progression 4, 11, 18, …, find (i) the 15th term and (ii) the sum of the first 15 terms.

(b) In the geometric progression 243, 81, 27, 9, …, find (i) the 8th term and (ii) the sum of the first 8 terms.

6. As part of her marathon training program, Kate has started running on the treadmill. If she runs 1 km the first day and increases her run by 0.5 km every week, in which week will she reach her goal of running 10 km per day?

7. In a certain city, the population has been projected to grow at a rate of 8% during each of the next 5 years. If the current population is 500,000, what is the expected population after 5 years?

6. Supplementary questions

1. Solve the following equations.

(a) $6x+7=2x-25$ (b) $\frac{1}{3}x-\frac{1}{4}x+5=11$

(c) $\frac{2x-3}{7x+2}=1$ (d) $2x^2-3x+1=0$

(e) $0.3x^2+2.1x-1.2=0$ (f) $\frac{1}{x}+\frac{1}{x+3}=4$

2. The concentration C (in mg/cc) of a drug in a person's bloodstream t hrs after injection is given by the equation

$$C=\frac{0.2t}{t^2+1}$$

Find the time when the concentration of the drug reaches 0.08 mg/cc.

3. Solve the following inequalities.

(a) $-3x+2>17$ (b) $-2<6x+1<\frac{1}{3}$

(c) $2x^2-128<0$ (d) $\frac{2x+5}{x-2}\geq 0$

(e) $|9x+1|\leq 46$ (f) $|7x-11|>24$

4. The diameter of a batch of ball bearings manufactured by BB Pte Ltd satisfies the inequality $|x - 0.1| \leq 0.01$, where x is the diameter in inches. What is the smallest diameter a ball bearing in the batch can have? What is the largest diameter a ball bearing in the batch can have?

5(a) In the arithmetic progression 27, 24, 21, …, find (i) the 11th term and (ii) the sum of the first 11 terms.

(b) In the geometric progression 2, –6, 18, –54 …, find (i) the 10th term and (ii) the sum of the first 10 terms.

6. Marcus has just received two job offers at two different companies. Company A offered an initial salary of $48,000 per year with guaranteed annual increases of $2,000 per year for the next 6 years. Company B offered an initial salary of $46,000 but will guaranteed that the salary will increase by 5% per year for the next 6 years.

(a) Which company is offering a higher salary for the 6th year of employment?

(b) Which company is offering more money for the first 6 years of employment?

BUSINESS MATHEMATICS

SESSION 3

FUNCTIONS AND THEIR GRAPHS I

At the end of the session, students should be able to:

1. Determine the equation of a straight line and sketch its graph on the x-y plane.
2. Understand the concept of a function and determine its domain and range.
3. Compute the sum, difference, product, quotient and composition of functions.

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1. Introduction

This session will discuss the fundamentals of coordinate geometry, which is essentially a system that allows us to represent relations between two variables in terms of ordered pairs of real numbers. Specifically, we will explore how variables which are related in a linear manner are represented on the plane. Next, the concept of mathematical functions will be discussed, together with ways in which we can perform operations on them.

Line graphs are used frequently in businesses and industry to display information in a visual manner for easy interpretation. The profit of a company may be dependent on the amount of sales. Or the cost of a parcel may depend on the weight of the parcel. In either case, we might need to illustrate the relation between the two variables on a graph. Working with graphs requires an understanding of the Cartesian coordinate system. The invention of the coordinate system was a pivotal point in the history of mathematics.

Likewise, functions are used to describe relations between two quantities. Determining these associations is useful in many aspects in real life, such as the exchange of foreign currencies to the examination of the speed of a chemical reaction. Expressing these relations as mathematical functions aids us in describing the relationships conveniently and unambiguously. The concept of a function is one of the most important concepts in mathematics.

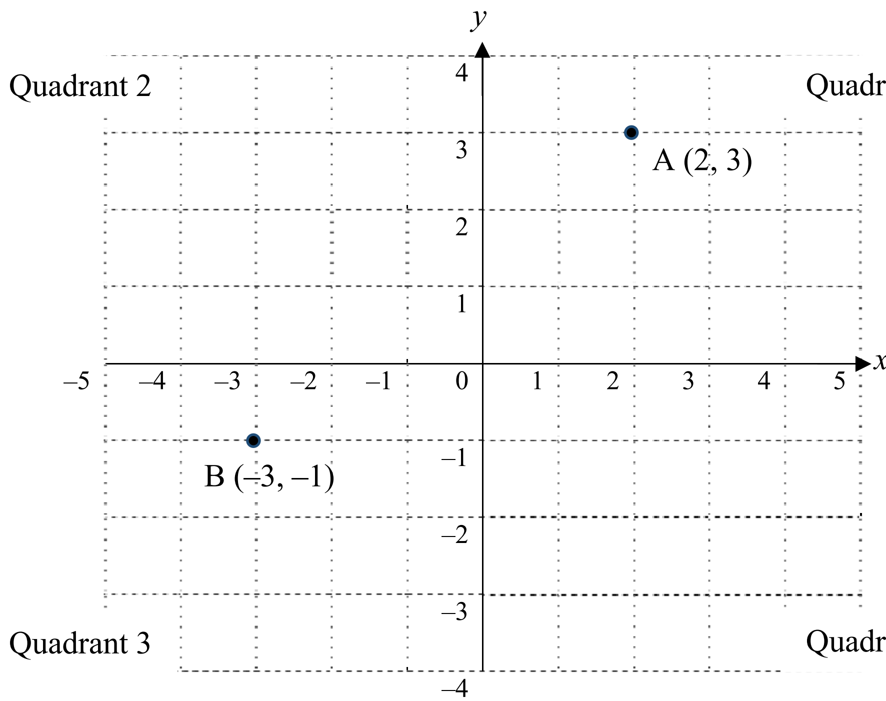
2. Equations and graphs of lines

2.1 Cartesian coordinate system

The Cartesian coordinate system consists of two perpendicular number lines. The horizontal line is the x-axis and the vertical line is the y-axis. The point where the two lines intersect is called the origin, O. The numbers on the axes to the right and above the origin are positive whereas the numbers on the axes to the left and below the origin are negative. The axes divide the plane into four parts: the first, second, third and fourth quadrant.

The location of a unique point in the coordinate system is indicated by an ordered pair in the form (x, y). The x value is always placed first, followed by the y value. The pair (x, y) is known as the coordinates of the point. Figure 3.1 illustrates the coordinate system with two points. Point A has coordinates (2, 3) and point B has coordinates (–3, –1).

Figure 3.1: The Cartesian coordinate system



2.2 Slope of straight lines

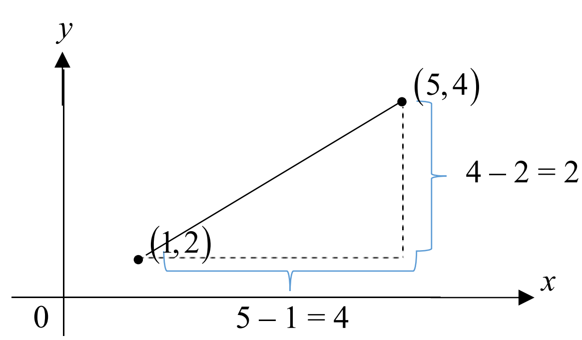
A useful concept when working with straight lines is slope or gradient, which is a measure of the "steepness" of the line. Formally, it is defined as the ratio of the vertical change to the horizontal change for any two points on the line, i.e. if (x_1, y_1) and (x_2, y_2) are two points on the line, then

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 3.1 Find the slope of the line passing through (1, 2) and (5, 4).

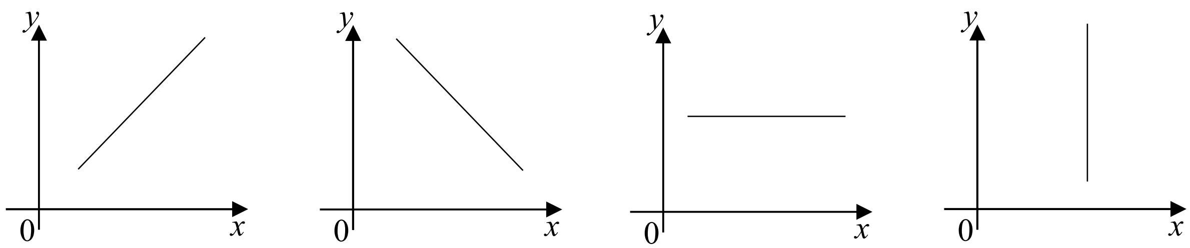
Solution

$$\begin{aligned}\text{Slope} &= \frac{4-2}{5-1} \\ &= \frac{2}{4} \\ &= \frac{1}{2}\end{aligned}$$



A line may have a positive slope, a negative slope, a zero slope or a slope which is undefined, as shown in Figure 3.2. A line with a positive slope slants upwards from left to right (as the value of x increases, the value of y increases) whereas a line with a negative slope slants downwards from left to right (as x increases, y decreases). A line with zero slope is a horizontal line (as x increases, y remains the same) and a line with a slope that is undefined is a vertical line (the value of x remains the same throughout). Note that in the case of a vertical line where the x values are constant, we cannot divide by zero in the computation of slope. Hence the slope is undefined.

Figure 3.2: Different types of slopes



Positive slope   Negative slope   Zero slope   Slope is undefined.

We can also use the slope of a line to determine whether two lines are parallel or perpendicular to each other. Two distinct lines are parallel when their slopes are equal or their slopes are undefined. Two distinct lines are perpendicular to each other when the product of their slopes is equal to –1. Let the slopes of the lines be m_1 and m_2. Figure 3.3 illustrates these properties of slopes.

Figure 3.3: Parallel and perpendicular lines

Parallel lines
$m_1 = m_2$

Perpendicular lines
$m_1 m_2 = -1$

Example 3.2 A line L_1 passes through the points (–1, 2) and (4, 12). Find the slope of the line which is (a) parallel to L_1, and (b) perpendicular to L_1.

Solution

(a) To find the slope of line parallel to L_1, we first find the slope of L_1.

$$\text{Slope of } L_1 = \frac{12-2}{4-(-1)} = 2$$

Since slopes are equal for parallel lines, slope of line parallel to $L_1 = 2$

(b) Let the slope of the line perpendicular to L_1 be m. Since the lines are perpendicular,

$$2m = -1$$

$$m = -\frac{1}{2}$$

2.3 Equation of straight lines

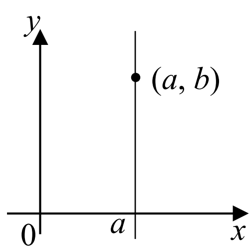
An equation of a straight line shows a linear relationship between two variables, usually denoted by x and y. Therefore, every straight line lying in the coordinate system can be represented by an equation. All the points with coordinates (x, y) that lie on the straight line will satisfy the equation.

Vertical and horizontal lines

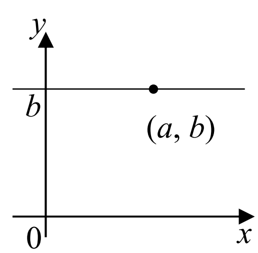
Firstly, consider the equation of a vertical line parallel to the y-axis that passes through a point with coordinates (a, b). Any other point on the line would have coordinates (a, y), where the x-coordinate is always a and y could be any real number. Thus the equation of a vertical line would be of the form $x = a$.

Next consider the equation of a horizontal line parallel to the x-axis that passes through a point with coordinates (a, b). Any other point on the line would have coordinates (x, b), where the y-coordinate is always b and x could be any real number. Thus the equation of a horizontal line would be of the form $y = b$. Figure 3.4 shows an example of these equations.

Figure 3.4: Equations of vertical and horizontal lines



Vertical line, $x = a$



Horizontal line, $y = b$

However, most lines are neither parallel to the x-axis nor to the y-axis but are usually inclined at an angle to the x-axis. We can use two different equations to express the relations between x and y for these lines, i.e. the point-slope form or the slope-intercept form.

Point-slope form

To determine the equation of a general line in point-slope form, we will first need to compute the slope of the line, m. Recall that if (x_1, y_1) and (x_2, y_2) are two fixed points on the line, then

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Now suppose that (x_1, y_1) is a fixed points on the line and (x, y) is a variable point on the line. Then using the formula for slope, we have $m = \dfrac{y - y_1}{x - x_1}$. Multiplying both sides by $(x - x_1)$, we obtain a formula that gives us the point-slope form for the equation of a straight line. Specifically, the point-slope form can be obtained from the formula

$$y - y_1 = m(x - x_1)$$

We can then re-arrange the equation such that all the terms are on the left hand side of the equation. This equation is called the point-slope form because it uses a given point (x_1, y_1) on the line and the slope, m, of the line.

Example 3.3 Find the equation of the line that passes through the points (3, 1) and (5, 11).

Solution

$$\begin{aligned}\text{Slope of line, } m &= \frac{11-1}{5-3}\\ &= \frac{10}{2}\\ &= 5\end{aligned}$$

To find equation of line:

$$\begin{aligned} y - 1 &= 5(x-3) \qquad [\text{Using } y - y_1 = m(x - x_1)]\\ y - 1 &= 5x - 15\\ 5x - y - 14 &= 0\end{aligned}$$

Slope-intercept form

The slope-intercept form of a line that has slope m and intersects the y-axis at the point $(0, b)$ is given by

$$y = mx + b$$

The number b is called the y-intercept of the line. This equation is called the slope-intercept form because it uses the slope of the line m and y-intercept of the line.

Example 3.4 The equation of a line is given by $2y = 3x - 8$. Find its slope and y-intercept.

Solution

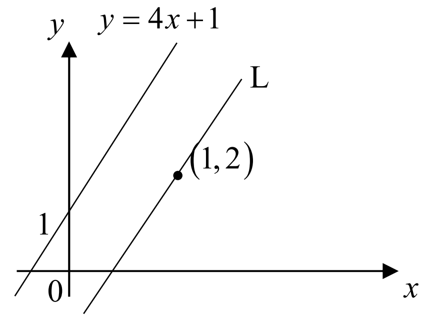
$$\begin{aligned} 2y &= 3x - 8\\ y &= \frac{3}{2}x - 4\end{aligned}$$

Slope = $\dfrac{3}{2}$, y-intercept = -4

Example 3.5 Find the equation of the line L that passes through the point (1, 2) and parallel to the line with equation $y = 4x + 1$.

Solution
As the lines are parallel, slope of L = 4

$$y-2=4(x-1) \qquad \text{[Using } y-y_1=m(x-x_1)\text{]}$$
$$y-2=4x-4$$
$$y=4x+2$$



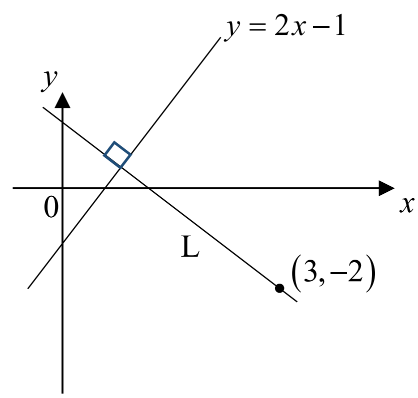
Example 3.6 Find the equation of the line L that passes through the point (3, –2) and is perpendicular to the line with equation $y = 2x – 1$.

Solution
Let the slope of the line be m. As the lines are perpendicular,

$$2m=-1$$
$$m=-\frac{1}{2}$$

Then

$$y-(-2)=-\frac{1}{2}(x-3) \quad \text{[Using } y-y_1=m(x-x_1)\text{]}$$
$$y+2=-\frac{1}{2}x+\frac{3}{2}$$
$$y=-\frac{1}{2}x-\frac{1}{2}$$

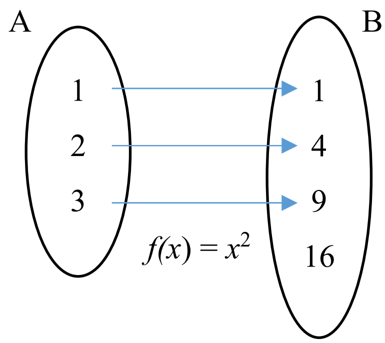


3. Functions

A function is a mathematical relation between two sets A and B such that each element in set A is associated with exactly one element in set B. The set A is called the domain of the function and the set B is called the codomain of the function. The mathematical relation is called the rule of the function and is usually denoted by f. If x is an element in set A, then the element in B that is associated with x is written $f(x)$. The set which contains all the elements of $f(x)$ is called the range of the function.

We can think of a function as a mapping between an element x in set A to an element $f(x)$ in set B. Figure 3.5 shows this mapping as well as the different components of a function.

Figure 3.5: An illustration of a function and its components



The domain is set A with numbers 1, 2 and 3.

The codomain is set B with numbers 1, 4, 9, and 16.

The rule of the function is given by $f(x) = x^2$.

The range is the set with numbers 1, 4 and 9.

In this course, the sets A and B will be the set of all real numbers. Note also that the element $f(x)$ associated with an element x is unique. This is especially important when we consider that many real life quantities are associated with only one value. For example, each item in a supermarket has only one selling price, although many items can have the same selling price.

When we evaluate a function, we are computing the value of $f(x)$ for a specific value of x. This is usually accomplished by simply replacing x with the particular value.

Example 3.7 The function f is defined by the rule $f(x) = x^2 - 3x + 1$. Find
(a) $f(2)$ (b) $f(-1)$ (c) $f(a)$ (d) $f(a + 1)$

Solution
(a) $f(2) = (2)^2 - 3(2) + 1 = -1$

(b) $f(-1) = (-1)^2 - 3(-1) + 1 = 5$

(c) $f(a) = (a)^2 - 3(a) + 1 = a^2 - 3a + 1$

(d) $f(a + 1) = (a + 1)^2 - 3(a + 1) + 1 = a^2 + 2a + 1 - 3a - 3 + 1 = a^2 - a - 1$

3.1 Domain of a function

In many practical applications, the domain of a function is restricted to certain values. For example, the price of an item should not be negative. However, if the nature of the application is not known, then we would need to find what restrictions should be placed on the domain. In general, it is understood that the domain consists of all values of x such that $f(x)$ is a real number. Specifically, we should take note of the following.

(i) Division by zero is not allowed.
(ii) The square root of a negative number is not a real number.

Example 3.8 Find the domain of each function.

(a) $f(x) = x^2 + x + 1$ (b) $f(x) = \dfrac{1}{x^2 - 9}$ (c) $f(x) = \sqrt{x - 2}$

Solution
(a) Substituting any real number x will obtain a real number for $f(x)$. Hence the domain for f is the set of all real numbers, i.e. $(-\infty, \infty)$ or $\mathbb{R}$.

(b) Factorising the denominator, we have $f(x) = \dfrac{1}{x^2 - 9} = \dfrac{1}{(x+3)(x-3)}$.

Substituting x as 3 or –3 will cause the denominator to be zero. As division by zero is undefined, the domain for f is the set of all real numbers excluding 3 and –3, i.e. $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

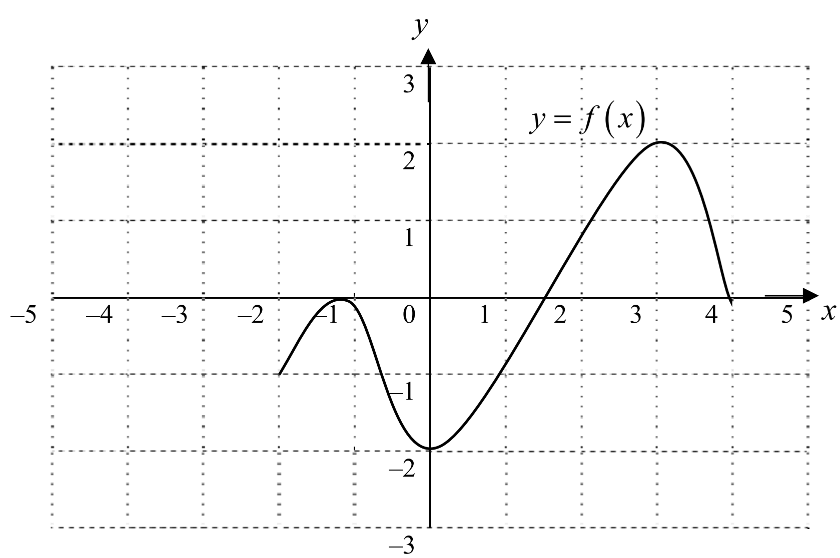
(c) As the square root of a negative number is not a real number, we will require that $x - 2$ be positive or zero, i.e. $x - 2 \geq 0$. Solving the inequality will give $x \geq 2$. Hence the domain for f is the set of all real numbers that are greater than or equal to 2, i.e. $[2, \infty)$.

3.2 Range of a function

Recall that the range of a function is the set of all elements that $f(x)$ can take. To determine the range of a function, we first note that as each number $f(x)$ corresponds to a value x in the domain of f, we can express the relation as an ordered pair $(x, f(x))$. Since ordered pairs correspond to points in a plane, we can thus present a function as a graph.

When a function is represented by a graph, the domain is the set of all real numbers lying on the x-axis and the range is the set of all real numbers lying on the y-axis.

Example 3.9 The graph of a function f is shown below.



(a) State the domain of the function.
(b) Find the value of $f(3)$.
(c) Determine the range of the function.

Solution

(a) Since x takes values between –2 and 4 inclusive, domain of f is [–2, 4].

(b) From the graph, $y = 2$ when $x = 3$. Hence we conclude that $f(3) = 2$.

(c) Since y takes values between –2 and 2 inclusive, range of f is [–2, 2].

We are able to obtain a lot of information about a function through a graph of the function. By plotting a few points $(x, f(x))$ of a function, we are usually able to determine the shape of the graph. For example, the graph of a linear equation is a straight line whereas the graph of a quadratic equation is a parabola. A function can also be defined by defined by giving different formulas for different sections of its domain. Such a function is called a piecewise-defined function.

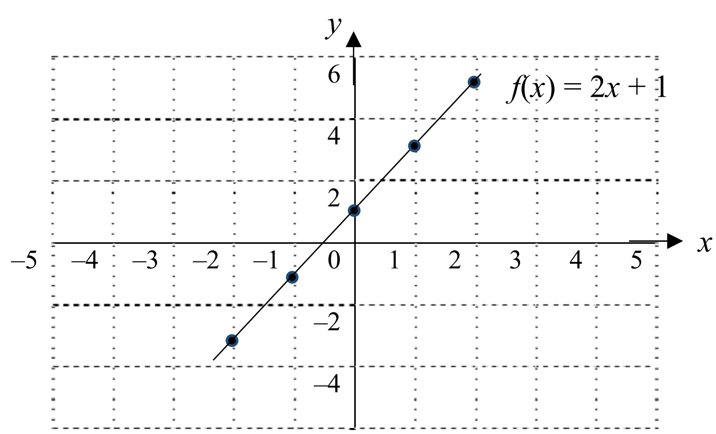
Example 3.10 Sketch the graph of the function defined by

(a) $f(x) = 2x + 1$ (b) $g(x) = x^2 + 1$

Solution

(a) We can plot a few points to determine the graph.

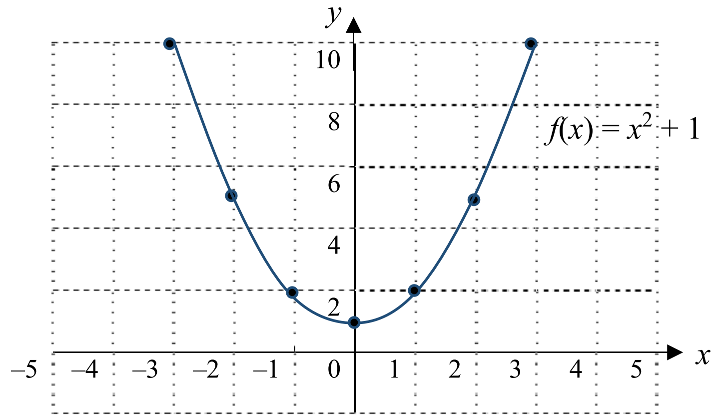
x	–2	–1	0	1	2
$f(x)$	–3	–1	1	3	5



The graph of $f(x) = 2x + 1$ is a straight line with gradient 2 and y-intercept 1.

(b) We can plot a few points to determine the graph.

x	–3	–2	–1	0	1	2	3
$f(x)$	10	5	2	1	2	5	10



The graph of $f(x) = x^2 + 1$ is a parabola y-intercept 1 and a minimum point at (0,1).

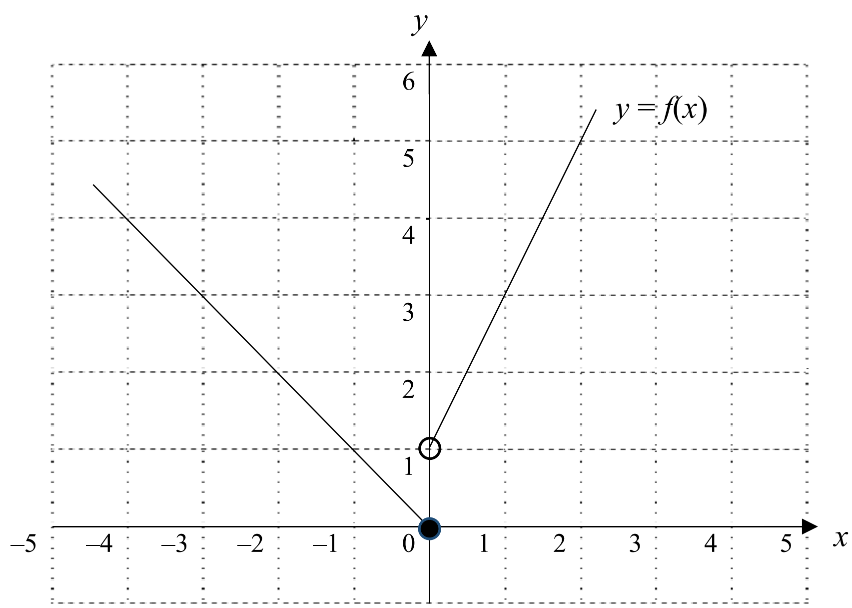
More systematic and sophisticated techniques for graphing linear and quadratic functions will be discussed in the next chapter.

Example 3.11 Sketch the graph of the function f defined by

$$f(x) = \begin{cases} -x & \text{if } x \le 0 \\ 2x+1 & \text{if } x > 0 \end{cases}$$

Hence, find the range of the function f.

Solution



When x is less than or equal to 0, i.e. the subdomain (–∞, 0), the rule for f is given by $f(x) = –x$. This is a linear equation with slope –1, y-intercept 0 and is represented graphically by a half-line. The solid circle at the coordinate (0, 0) indicates that the subdomain includes 0.

When x is more than 0, i.e. the subdomain $(0, \infty)$, the rule for f is given by $f(x) = 2x + 1$. This is a linear equation with slope 2, y-intercept 1 and is represented graphically by a half-line. The hollow circle at the coordinate (0, 1) indicates that the subdomain does not include the number 0.

From the graph, range of $f = [0, \infty)$.

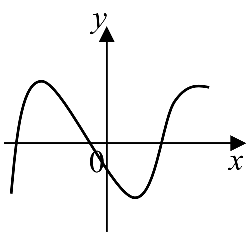
3.3 Vertical line test for a function

Recall that in a function the element $f(x)$ associated with the element x is unique. In other words, for any value of x, there can only be one value of $f(x)$. This definition gives us a method for determining whether a curve is the graph of a function. We can easily check whether a value for $f(x)$ is unique by drawing vertical lines through the graph. If the value of $f(x)$ is unique, the vertical line should intersect the graph at only one point.

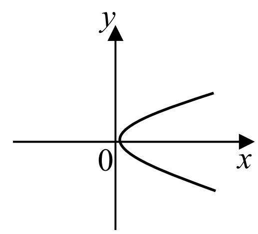
Essentially, a curve in the x-y plane is a function if and only if each vertical line intersects it in at most one point.

Example 3.12 Determine which of the curves shown below are graphs of functions.

(a)



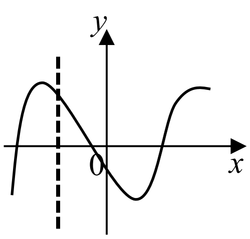
(b)



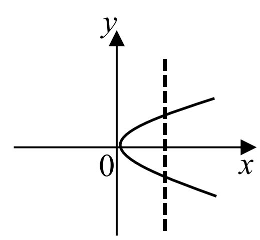
Solution

The curve in (a) is a function since each vertical line intersects the curve in at most one point. The curve in (b) is not a function as a vertical line intersects the curve at 2 points.

(a)



(b)



3.4 Algebraic operations on functions

Given two functions f and g, we can perform algebraic operations on the functions that are similar to operations on polynomials. The sum $f + g$, the difference $f - g$ and the product fg are defined as follows.

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$

The quotient f/g is defined in an analogous manner, with the additional constraint that the domain of f/g is adjusted to exclude all values x such that $g(x) = 0$.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example 3.13 Let $f(x) = x^2 - 1$ and $g(x) = x - 2$. Find the sum, difference, product and quotient of the functions f and g.

Solution

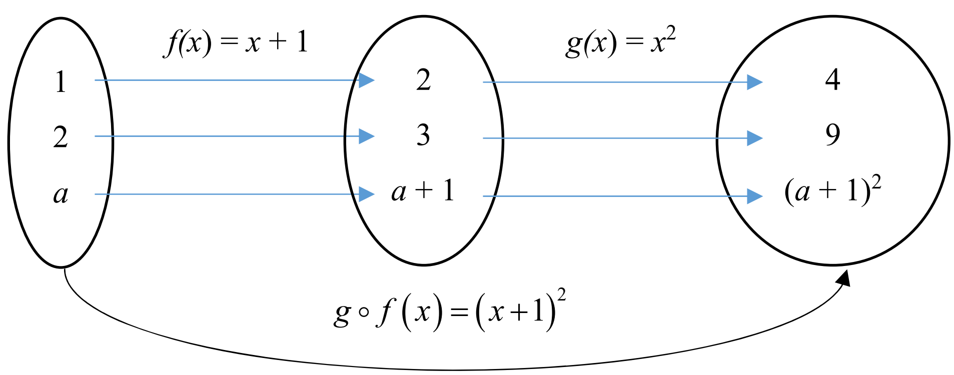
$$(f+g)(x) = f(x) + g(x) = (x^2+1) + (x-2) = x^2 + x - 1$$
$$(f-g)(x) = f(x) - g(x) = (x^2+1) - (x-2) = x^2 - x + 3$$
$$(fg)(x) = f(x)g(x) = (x^2+1)(x-2)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2+1}{x-2}$$

Composition of two functions

One way to build up a more complicated function from simpler functions is through a composition of functions. This is achieved by first evaluating a function f at the value x to obtain $f(x)$. Next, the function g is evaluated at the value $f(x)$. The resultant function is known as the composite function of g and f and is denoted by $g \circ f$.

For example, suppose the functions f and g are defined by $f(x) = x + 1$ and $g(x) = x^2$. Figure 3.6 shows the interpretation of the composite function $g \circ f$.

Figure 3.6: The interpretation of a composite function



In general, if f and g are functions, then the composition of g and f is defined by the function

$$(g \circ f)(x) = g[f(x)]$$

To determine the domain of $g \circ f$, observe that values of x must lie in the domain of f since we are applying the function f to x. As we are applying the function g to values of $f(x)$, $f(x)$ must also lie in the domain of g. Hence, the domain of $g \circ f$ is the set of all x in the domain of f such that $f(x)$ lies in the domain of g.

An additional point to note is that a composite function depends on the function that we "apply first". Therefore, the composite function $g \circ f$ is usually different from the function $f \circ g$.

Example 3.14 Let $f(x) = x^2$ and $g(x) = \sqrt{x} - 1$. Find the composite functions $g \circ f$ and $f \circ g$.

Solution

$$\begin{aligned}(g \circ f)(x) &= g\left[f(x)\right] \\ &= g\left(x^2\right) \\ &= \sqrt{x^2} - 1 \\ &= x - 1\end{aligned}$$

$$\begin{aligned}(f \circ g)(x) &= f\left[g(x)\right] \\ &= f\left(\sqrt{x} - 1\right) \\ &= \left(\sqrt{x} - 1\right)^2 \\ &= x - 2\sqrt{x} + 1\end{aligned}$$

4. Discussion questions

1. Find the slope of the line passing through the following points.

(a) $(-2,1)$ and $(3,-5)$ (b) $(2,3)$ and $(-6,1)$

2. Find the equation of the line that passes through the following points.

(a) $(5,-1)$ and $(5,8)$ (b) $(-1,3)$ and $(3,11)$

3. Find the equation of the line that has y-intercept 2 and is parallel to the line with equation $y = 4x - 7$.

4. Find the equation of the line that passes through the point (3, –1) and is perpendicular to the line that passing through the points (–2, –1) and (4, 3).

5. The function f is defined by $f(x) = 3^x + 1$. Find the value of $f(1)$, $f(0)$ and $f(-1)$.

6. Determine the domain of the following functions.

(a) $f(x) = x^2 + x + 3$ (b) $f(x) = \sqrt{5-x}$

(c) $g(x) = \dfrac{1}{x^2 - 2x - 3}$ (d) $g(x) = \dfrac{\sqrt{1-x}}{x^2 - 4}$

7. Sketch the graph of the function f defined by $f(x) = \begin{cases} 2x & \text{if } x \le 0 \\ x+3 & \text{if } x > 0 \end{cases}$.

Hence, find the range of the function f.

8. The functions f and g are defined by the rules $f(x) = x^2 + 2x - 1$ and $g(x) = x + 1$. Find the rules for $f+g$, $f-g$, fg, $\dfrac{f}{g}$.

9. The functions f and g are defined by the rules $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$. Find the composite functions $g \circ f$ and $f \circ g$.

5. Supplementary questions

1. Find the slope of the line passing through the following points.

(a) $(-1,-1)$ and $(5,11)$ (b) $(1,2)$ and $(3,2)$

2. Find the equation of the line that passes through the following points.

(a) $(4,3)$ and $(6,-1)$ (b) $(3,-5)$ and $(3,17)$

3. Find the equation of the line that has y-intercept 1 and is perpendicular to the line with equation $y = 2x + 11$.

4. Find the equation of the line that passes through the point (1, 5) and is parallel to the line that passing through the points (–5, –1) and (–2, 7).

5. The function f is defined by $f(x) = (2^x + 1)^2$. Find the value of $f(3)$ and $f(-2)$.

6. Determine the domain of the following functions.

(a) $f(x) = x^3 - x^2$ (b) $f(x) = \sqrt{1-2x}$

(c) $g(x) = \dfrac{1}{x^2 + 2x + 1}$ (d) $g(x) = \dfrac{x-3}{\sqrt{(x-1)(x+1)}}$

7. Sketch the graph of the function f defined by $f(x) = \begin{cases} 3x & \text{if } x \le 0 \\ -x+2 & \text{if } x > 0 \end{cases}$.

Hence, find the range of the function f.

8. The functions f and g are defined by the rules $f(x) = -2x^2 + 11$ and $g(x) = x^2 - 4$. Find the rules for $f+g$, $f-g$, fg, $\dfrac{f}{g}$.

9. The functions f and g are defined by $f(x) = 3^{x-1}$ and $g(x) = 2x + 1$. Find the composite functions $g \circ f$ and $f \circ g$.

BUSINESS MATHEMATICS

SESSION 4

FUNCTIONS AND THEIR GRAPHS II

At the end of the session, students should be able to:

1. Sketch graphs of a linear and quadratic functions.
2. Apply linear functions to linear depreciation, cost, revenue and profit functions and break-even analysis.
3. Apply quadratic functions to demand and supply curves and market equilibrium quantities.
4. Understand the use of functions in mathematical modelling.

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1. Introduction

This session will extend the discussion on functions by first using systematic techniques to express linear and quadratic functions visually on the *x*-*y* plane and subsequently exploring the manners in which these functions are applied in business and economics.

Linear functions play a significant role in quantitative business and economics analysis. This is mainly due to two reasons: 1) Many quantities in these fields are linear in nature or are linear within a certain interval. These quantities can thus be framed in terms of linear functions. 2) Linear functions are relatively easier to work with. Thus, many problems are formulated by assuming that the real life quantity follows a linear model and in many cases, these assumptions are justified. Some examples of these models are linear depreciation and the cost, revenue and profit functions of a business.

When a business or economic quantity reaches a maximum or minimum level at some point, a linear function might not be sufficient to model this quantity. Rather, a quadratic function would be more suitable to interpret the relationship. A quadratic function is chiefly defined by its vertex (maximum or minimum point) and many problems that are formulated assumes that the quantity is characterised by a peak or bottom. Some examples of these are demand and supply curves in a market economy.

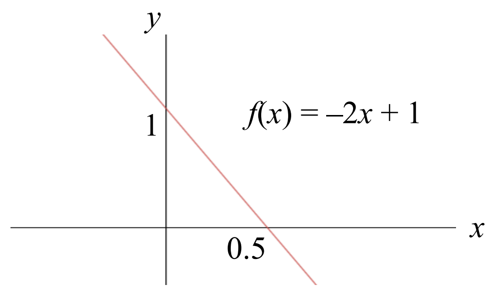
2. Graphs of a linear and quadratic function

2.1 Linear functions

A linear function is defined by $f(x) = mx + b$, where m and b are constants. The value m is the slope of the function and b is the y-intercept of the function. The graph of a linear function is a straight line. Since the graph is a straight line, we only require two points to sketch the graph.

Example 4.1 Sketch the graph of $f(x) = -2x + 1$.

Solution
The y-intercept (where the graph cuts the y-axis) is 1, i.e. when $x = 0$, $y = f(0) = 1$.
When $y = f(x) = 0$, $-2x + 1 = 0 \Rightarrow x = 0.5$
Hence, the x-intercept (where the graph cuts the x-axis) is 0.5.
Graph:



2.2 Quadratic functions

A quadratic function is defined by $f(x) = ax^2 + bx + c$, where a, b and c are constants, $a \neq 0$. The graph of a quadratic function is a parabola. The following properties will aid us in sketching the curve of a quadratic function.

1. The domain of f is the set of all real numbers.
2. The graph of f is a parabola. If $a > 0$, the parabola opens upwards, i.e. $\cup$ shape. If $a < 0$, the parabola opens downwards, i.e. $\cap$ shape.
3. The vertex (highest or lowest point) of the parabola is at the point $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.
4. The parabola is symmetrical about the line $x = -\dfrac{b}{2a}$. The line $x = -\dfrac{b}{2a}$ is also known as its axis of symmetry.
5. The y-intercept is the value of $f(0)$ or c. The x-intercepts (if any) can be found by solving the equation $f(x) = 0$.

Example 4.2 Sketch the graph of $f(x) = -2x^2 + 3x - 1$.

Solution
Comparing $f(x) = -2x^2 + 3x - 1$ to the general form, we observe that $a = -2$, $b = 3$ and $c = -1$. The graph is a parabola. Since $a = -2 < 0$, the parabola opens downwards, i.e. the shape is $\cap$.

To obtain the vertex:

$$x = -\frac{b}{2a} = -\frac{3}{2(-2)} = \frac{3}{4}$$

$$y = f\left(-\frac{b}{2a}\right) = f\left(\frac{3}{4}\right) = -2\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right) - 1 = \frac{1}{8}$$

Hence, vertex $= \left(\frac{3}{4}, \frac{1}{8}\right)$

To obtain the intercepts:

$$y - \text{intercept} = f(0) = -2(0)^2 + 3(0) - 1 = -1$$

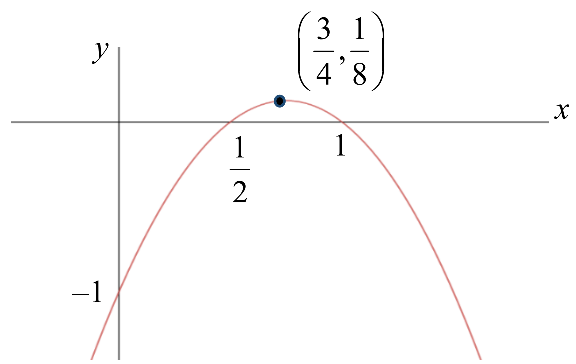
To compute the x-intercepts (if any), equate $y = f(x) = 0$

$$-2x^2 + 3x - 1 = 0$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(-2)(-1)}}{2(-2)}$$

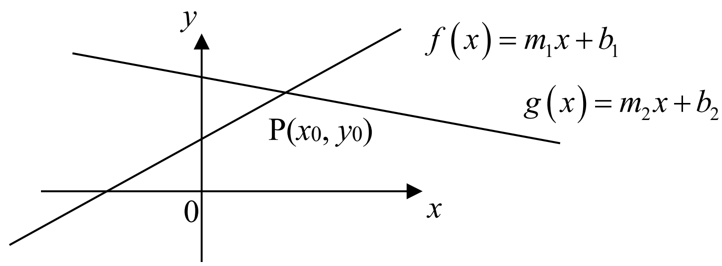
$$= 1 \ \ or \ \ \frac{1}{2}$$

Graph:



2.3 Intersection of two graphs

At times, the solution of certain practical problems requires finding the point of intersection of two graphs. Solving these problems algebraically usually requires us to solve a system of equations. As an example, suppose there are two linear functions $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$ and that they intersect at a point $P(x_0, y_0)$.



At the point P, the coordinates satisfy both the functions f and g, i.e. $f(x_0) = g(x_0)$. Hence we can obtain the value of x_0 by equating the functions and solving the equation $m_1x_0 + b_1 = m_2x_0 + b_2$. The corresponding value of y_0 can be computed from either function.

Example 4.3 Find the point of intersection of the graphs of $f(x) = x + 1$ and $g(x) = -2x + 7$.

Solution
At the point of intersection, $f(x) = g(x)$. Hence

$$\begin{aligned} x+1 &= -2x+7 \\ 3x &= 6 \\ x &= 2 \\ f(2) &= 2+1 = 3 \end{aligned}$$

Alternatively, $g(2) = -2(2)+7 = 3$. In either case, point of intersection = (2, 3)

3. Applications of linear functions

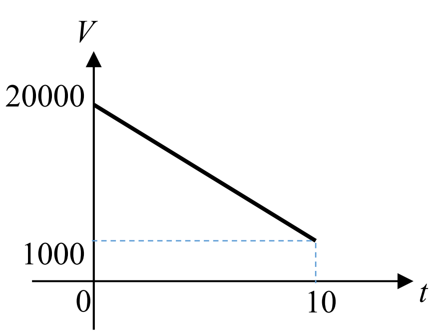
3.1 Linear depreciation

Certain assets such as furniture, machines, buildings and vehicles owned by businesses depreciate in value over time and firms are legally allowed to incorporate depreciation into their tax returns. Linear depreciation is often used for this purpose. The following example illustrates a manner to describe the book value of an asset that is being depreciated mathematically.

Example 4.4 A computer system has an original value of \$20,000 and is to be depreciated linearly over 10 years. The scrap value of the system is estimated to be at \$1,000.

(a) Find an expression giving the book value of the computer system at the end of year t.
(b) What is the book value of the computer system at the end of the third year?
(c) What is the rate of depreciation of the computer system?

Solution
(a) Let $V(t)$ be the book value of the computer system at the end of year t. Since the system depreciates linearly, $V(t)$ is a linear function. Note that the original value of the system is \$20,000 i.e. when $t = 0$, $V(0) = 20{,}000$. The value of the system is \$1,000 when $t = 10$, i.e. $V(10) = 1{,}000$. This gives us two points (0, 20000) and (10, 1000), allowing us to sketch the graph. The required expression is the equation of the graph.



Slope of line $m = \dfrac{1000-20000}{10-0} = -1900$

Using the point-slope form of an equation of line,

$$\begin{aligned} V - 20000 &= -1900(t-0) \\ V &= -1900t + 20000 \end{aligned}$$

Hence the required expression is

$$V(t) = -1{,}900t + 20{,}000$$

(b) The book value at the end of the third year is given by $V(3)$.

$$\begin{aligned} V(3) &= -1900(3) + 20000 \\ &= \$14,300 \end{aligned}$$

(c) The slope of the graph is –1900 which indicates that the system is decreasing in value at a rate of \$1,900 per year. Hence the rate of depreciation is \$1,900 per year.

3.2 Linear cost, revenue, profit functions

In any form of business, the owner or chief executive officer must constantly keep track of their operating costs and revenue that results from their sale of products or services. More importantly, the management must have the means to measure the profits realised on a daily, weekly or monthly basis. Three linear functions provide the tools to do so, namely the total cost function, the revenue function and the profit function.

Suppose x denotes the number of units of a product manufactured or sold. The total cost function gives the total cost of manufacturing x units of the product. The costs that are incurred from operating a business can be categorized into fixed costs and variable costs. Fixed costs are costs that remain constant regardless of the number of sales and examples of fixed costs include rental fees, executive salaries, utility fees and insurance. Variable costs are costs that vary with the production or sales and examples of variable costs are costs for raw materials and wages for labour. Thus, if a firm has a fixed cost of F dollars and a production cost of c dollars per unit, the total cost function $C(x)$ is given by

$$C(x) = cx + F$$

The revenue function gives the total revenue realised from the sale of x units of the product. If the firm has a selling price of s dollars per unit, the total revenue function $R(x)$ is given by

$$R(x) = sx$$

The profit function gives the total profit realised from manufacturing and selling x units of the product. As profit is usually measured by the difference between the total revenue and the total cost, the total profit function $P(x)$, is given by

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= sx - cx - F \\ &= (s - c)x - F \end{aligned}$$

Realistically, the cost, revenue and profit functions need not be linear in nature and these nonlinear functions are best studied with the aid of calculus. However, as linear cost, revenue and profit functions do arise or can be approximated in business, it would be sensible to understand these tools and the insights they give into business operations.

Example 4.5 Fanz Pte Ltd manufactures electric fans at a production cost of \$25 per unit. It has a monthly fixed cost of \$20,000 and sells its fans at a price of \$100 per unit. Find its cost function, revenue function and profit function.

Solution

Let x be the number of units produced and sold. Then

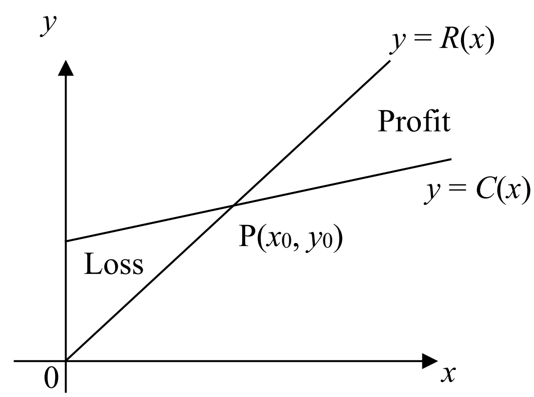
$$\begin{aligned} C(x) &= 25x + 20{,}000 \\ R(x) &= 100x \\ P(x) &= R(x) - C(x) \\ &= 100x - (25x + 20{,}000) \\ &= 75x - 20{,}000 \end{aligned}$$

Break-even analysis

A concept that is of significant value to businesses is the break-even level. The break-even level of operations is defined as the level of production at which the firm neither makes a profit nor sustains a loss. At that level, the quantity of production is known as the break-even quantity and the revenue realised is known as the break-even revenue.

Figure 4.1 illustrates visually the concept of the break-even level. When the total cost and total revenue functions are graphed, the point of intersection, P, between the two functions is the break-even point. The break-even quantity is x_0 and the break-even revenue is y_0. Any production level above the break-even quantity will result in a profit for the business. Any production level below the break-even quantity will result in a loss for the business.

Figure 4.1: A visual look at the break-even level



Since the break-even point is the intersection of two functions, we can compute the break-even quantity by solving the system of equations, i.e. $R(x) = C(x)$. Alternatively, as the firm neither makes a profit nor loss, we can also compute the break-even quantity by assuming the profit to be zero, i.e. $P(x) = 0$. Once the break-even quantity is found, we can determine the break-even revenue by substituting the break-even quantity into the revenue function.

Example 4.6 Pretz Inc. manufactures its products at a cost of \$5 per unit and sells them for \$13 per unit. Assuming a monthly fixed cost of \$12,000, determine the break-even point.

Solution
Let x be the number of units produced and sold. Then $C(x) = 5x + 12{,}000$ and $R(x) = 13x$.
At the break-even point, $R(x) = C(x)$,

$$\begin{aligned} 13x &= 5x + 12{,}000 \\ 8x &= 12{,}000 \\ x &= 1{,}500 \end{aligned}$$

Alternatively, $P(x) = 13x - (5x + 12{,}000) = 8x - 12{,}000$.
At the break-even point $P(x) = 0$,

$$\begin{aligned} 8x - 12{,}000 &= 0 \\ 8x &= 12{,}000 \\ x &= 1{,}500 \end{aligned}$$

Substituting $x = 1500$ into the revenue function, $R(1{,}500) = 13(1{,}500) = 19{,}500$
Therefore break-even quantity = 1500 and break-even revenue = \$19,500.

Example 4.7 The management of Tony's Thermostats must decide between two manufacturing processes for its new model of thermostat. The monthly costs for the two processes are summarised below.

Process 1: Monthly cost, $C_1(x) = 30x + 10{,}000$
Process 2: Monthly cost, $C_2(x) = 25x + 20{,}000$

where x is the number of thermostats produced. If the projected monthly sales are 600 thermostats at a unit price of \$50, which process should the management choose to maximize the firm's profits?

Solution
In both processes, $R(x) = 50x$.
Break-even level of operation using the first process is obtained by

$$\begin{aligned} 50x &= 30x + 10{,}000 \\ 20x &= 10{,}000 \\ x &= 500 \end{aligned}$$

Break-even level of operation using the second process is obtained by

$$\begin{aligned} 50x &= 25x + 20{,}000 \\ 25x &= 20{,}000 \\ x &= 800 \end{aligned}$$

Since the projected monthly sales is 600, the management should choose the first process which will give the company a profit.

4. Applications of quadratic functions

4.1 Demand and supply curves

In a free-market economy, consumer demand for a particular commodity depends on the its unit price. The relationship between a commodity's unit price and the quantity demanded is known as the demand function and its graph is known as a demand curve.

Mathematically, a demand function is often denoted by $p = D(x)$, where p measures the unit price and x measures the number of units of commodity. As the demand of a commodity increases, the unit price typically decreases. Hence, the demand function is usually a decreasing function, i.e. as x increases, p decreases. Furthermore, since p and x are always positive, the demand curve lies in the first quadrant only.

Example 4.8 The demand function for a brand of mini-speakers is given by

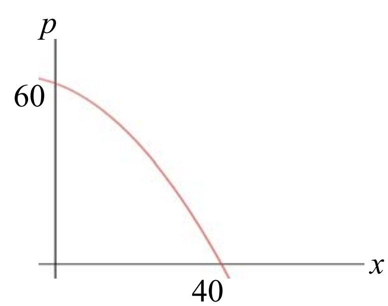
$$p = -0.025x^2 - 0.5x + 60$$

where p is the wholesale unit price in dollars and x is the quantity demanded each month, measured in units of a hundred.

(a) Sketch the corresponding demand curve.
(b) What is the maximum quantity demanded per month?
(c) Above what price will there be no demand?

Solution

(a) The curve can be sketched using the techniques in Section 2.2 in this chapter. However, as the values of x and p are only positive, the curve is truncated and only the portion in the first quadrant need to be shown.



(b) The maximum quantity demanded occurs when the price is zero, that is $p = 0$.

$$-0.025x^2 - 0.5x + 60 = 0$$

$$x = \frac{-(-0.5) \pm \sqrt{(-0.5)^2 - 4(-0.025)(60)}}{2(-0.025)}$$

$$= 40 \quad or \quad -60(rej)$$

Since x must be non-negative, we reject $x = -60$. Hence, maximum quantity demanded per month is 4,000 (x is measured in units of a hundred).

(c) When there is no demand, $x = 0$. Thus the price above which there is no demand is \$60.

In the same free-market economy, the unit price of a commodity is also dependent on the availability of the commodity. The relationship between a commodity's unit price and the quantity supplied in the market is known as the supply function and its graph is known as a supply curve.

Mathematically, a supply function is also denoted by $p = S(x)$, where p measures the unit price and x measures the number of units of commodity. As the unit price of a commodity increases, it will typically induce the supplier to increase the supply of the commodity in the market. Hence, the supply function is usually an increasing function, i.e. as x increases, p increases. Similar to the demand curve, since p and x are always positive, the supply curve lies in the first quadrant only.

Example 4.9 The supply function for a brand of mini-speakers is given by

$$p = 0.02x^2 + 0.6x + 20$$

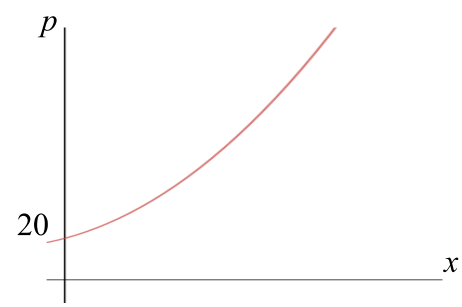
where p is the wholesale unit price in dollars and x is the quantity, measured in units of a hundred, that will be made available in the market by the supplier.

(a) Sketch the corresponding supply curve.

(b) What is the lowest price at which the supplier will make the mini-speakers available to the market?

Solution

(a) The curve can be sketched using the techniques in Section 2.2 in this chapter. Similar to the demand curve, the curve is truncated and only the portion in the first quadrant is shown.



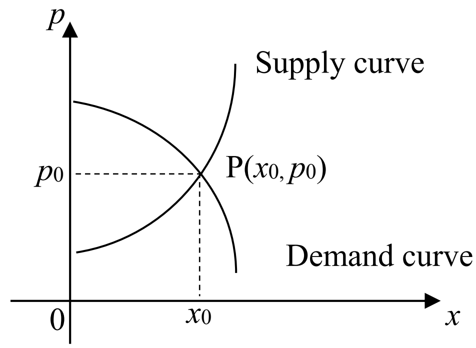
(b) The lowest price occurs when there is no supply to the market, i.e. $x = 0$. Thus the lowest price is \$20.

4.2 Market equilibrium

Under the circumstances of pure competition, the price of a commodity will eventually settle at a fixed level. Why is this so? If the price of a commodity is too high, the consumer will not purchase the commodity. However, if the price is too low, the supplier will not produce. At this price level, the supply of the commodity will be equal to the demand for it. We say that market equilibrium prevails when the quantity produced is equal to the quantity demanded. The quantity produced at market equilibrium is called the equilibrium quantity and the corresponding price is called the equilibrium price.

By the definition of market equilibrium, we note that market equilibrium corresponds to the point at which the demand and supply curve intersect. Figure 4.2 shows the graphs of a demand curve and a supply curve on the same axis. The intersection of both curves, P, corresponds to the point where market equilibrium occurs. At this intersection, x_0 is the equilibrium quantity and p_0 is the equilibrium price. In order to determine the equilibrium quantity and price, we will have to solve the demand and supply functions simultaneously.

Figure 4.2: Market equilibrium occurs where the demand and supply curve intersect



Example 4.10 The demand function for a brand of mini-speakers is given by

$$p = D(x) = -0.025x^2 - 0.5x + 60$$

and the corresponding supply function is given by

$$p = S(x) = 0.02x^2 + 0.6x + 20$$

where p is in dollars and x is measured in units of a hundred. Find the equilibrium quantity and price.

Solution

At market equilibrium, the price level is fixed. Hence to determine the equilibrium quantity, we equate the demand and supply functions.

$$-0.025x^2 - 0.5x + 60 = 0.02x^2 + 0.6x + 20$$

$$0.045x^2 + 1.1x - 40 = 0$$

$$x = \frac{-1.1 \pm \sqrt{(1.1)^2 - 4(0.045)(-40)}}{2(0.045)}$$

$$= 20 \quad or \quad -\frac{400}{9}(rej)$$

To obtain the equilibrium price, we substitute $x = 20$ into either the demand or supply function.

$$p = 0.02(20)^2 + 0.6(20) + 20$$

$$= 40$$

Thus the equilibrium quantity is 2,000 and the corresponding equilibrium price is \$40.

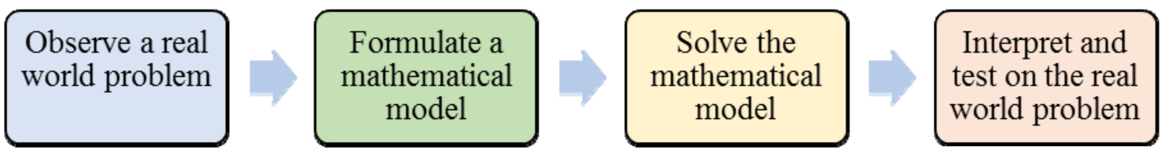
5. Functions and mathematical models

5.1 Mathematical modelling

In the real world, mathematics is often used to solve problems in many fields, such as business, economics, social, life and physical sciences. We have already explored some of these problems in the previous sections but mathematics is used far more extensively than what is presented earlier. A few more examples are the intensity of air pollutants, the quality of wine, the value of a stock and the projected costs in government expenditure.

Regardless of the field, the real world problem is analysed using a process called mathematical modelling. Figure 4.3 highlights the key steps in this process.

Figure 4.3: Key processes in mathematical modelling



The first step in mathematical modelling is to observe a real world problem that we would like to solve. For example, we might like to know how much money we would receive at the end of a certain number of years if we deposit \$1000 into the bank every month. Or we might like to predict the gross revenue of the gaming industry in the next 5 years. We would need to recognise clearly the issues that we want to resolve.

The second step involves formulating the problem in the language of mathematics. This requires the identification of the different variables involved as well as the use of various mathematical techniques to form a model. For example, we might find that the amount of money accrued in a bank account depends on the interest rate, the number of years and the amount of monthly deposit. A mathematical model might then be formed using algebraic techniques.

The third step entails us using the appropriate mathematical tools to solve the model that we have constructed. These tools are presented throughout this book, ranging from numerical operations to calculus.

The final step requires us to interpret and test the results that we obtained in the context of the original real-world problem. Although some of these results will describe the situation accurately, at most times, it will provide only an approximation to the solution that we want to achieve. To test the accuracy of the model, we would need to observe how well it describes the real-world situation and how well it predicts past and/or future behaviour. If the results are unsatisfactory, we would have to return to the second step and reconsider the model or its assumptions.

As many of these problems are modelled by functions, we will recall some of these functions and how they are used to construct models.

5.2 Functions

Polynomial functions
A polynomial function of degree n is of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

where n is a non-negative integer and the coefficients a_0, a_1, …, a_n are constants.

A polynomial function of degree 1 is a linear function and a polynomial of degree 2 is a quadratic function. An example of a linear function is $f(x) = 2x - 7$ and an example of a quadratic function is $g(x) = 3x^2 + x - 17$. Note that a polynomial function is always defined for all values of x. Hence the domain for any polynomial function $(-\infty, \infty)$.

Rational functions
A rational function has the form $R(x) = \dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomial functions. An example of a rational function is $R(x) = \dfrac{-x+11}{x^3+2x-5}$. Since division by zero is not allowed, the domain of a rational function is the set of all real numbers except where $g(x) = 0$.

Power functions
A power function has the form $f(x) = x^r$ where r is any real number. An example of a power function is $f(x) = x^{0.5}$.

Many functions that are used as mathematical models are a combination of these types of functions. In the next section, we will examine these functions in greater detail when we try to construct some models to describe real-world situations.

5.3 Constructing mathematical models

There are two main ways that we can construct mathematical models. We can consider the problem from a theoretical angle and think of an equation to describe the situation or we can study past data and observe any patterns that arise from it.

Construction from theory
The following guidelines can be used to construct models from algebraic arguments.

1. Assign a letter to each variable in the problem. At times, it might be useful to draw and label a diagram.
2. Find an expression for the quantity that you want.
3. Use any conditions given to write the quantity that you want as a function f of one variable. Note any restrictions placed on the domain of f due to physical considerations.

Example 4.11 NZT charges \$300 per person for a tour of an exclusive island if exactly 200 people sign up for the tour. However, if more than 200 people sign up for the tour, then each fare is reduced by \$1 for each additional person. Assuming that more than 200 people sign up for the tour, find a function giving the revenue realised by the NZT.

Solution
Since the revenue depends on the number of people above 200, we will assign the letter x to this variable, i.e. let x be the number of people above 200 who sign up for the tour.

The number of people who sign up for the tour would thus be $200 + x$. As each fare will be reduced by \$1 per additional person, the fare per person will be $300 - x$. The revenue would be the number of people multiplied by the fare per person.

$$\begin{aligned} R(x) &= (200+x)(300-x) \\ &= -x^2 + 100x + 60,000 \end{aligned}$$

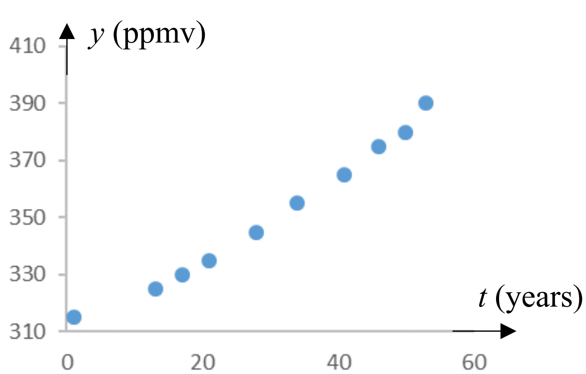
The least possible fare is \$0. Hence $300 - x \geq 0$ or $x \leq 300$. Since x cannot be negative as well, the required function is thus $R(x) = -x^2 + 100x + 60,000$ with domain [0, 300].

Construction from past data
If we have past data points for the required variable, we might be able to construct a scatterplot depicting the relation between two variables on an x-y plane. By fitting a curve onto all the points on the scatterplot, we can estimate the equation of the curve using mathematics. Nowadays, the estimated equation is usually determined with the aid of technological tools.

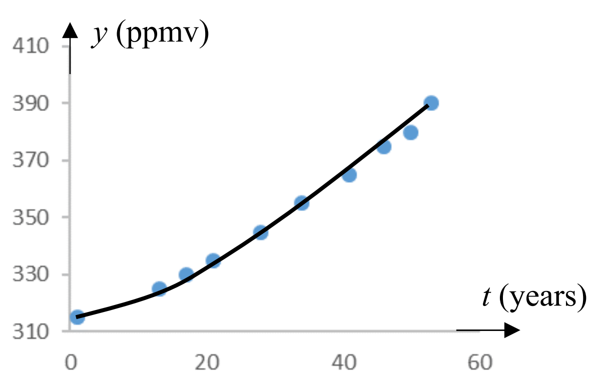
Example 4.12 One of the major causes of global warming is the increase of carbon dioxide (CO_2) levels in the atmosphere. The Keeling curve gives the average amount of CO_2, measured in parts per million volume (ppmv), in the atmosphere from 1958 to 2010. Some randomly selected data points are shown in the table below.

Year	1958	1970	1974	1978	1985	1991	1998	2003	2007	2010
Amount	315	325	330	335	345	355	365	375	380	390

Using the data points, a scatter plot is drawn and a curve is fitted to the points.



Scatterplot



Curve of best fit

The mathematical equation relating the approximate amount of CO_2 in the atmosphere is estimated to be $A(t) = 0.012313t^2 + 0.7545t + 313.9$, $(1 \leq t \leq 53)$, where t is measured in years, with $t = 1$ corresponding to 1958.

(a) Use the model to estimate the average amount of CO_2 in the atmosphere in 1980.

(b) Using the model, predict the average amount of CO_2 in the atmosphere in 2025.

Solution

(a) In the year 1980, $t = 23$.

$$A(23) = 0.012313(23)^2 + 0.7545(23) + 313.9 = 337.77$$

Hence the average amount of CO_2 in the atmosphere is approximately 337.77 ppmv.

(b) In the year 2025, $t = 68$.

$$A(68) = 0.012313(68)^2 + 0.7545(68) + 313.9 = 422.14$$

The predicted average amount of CO_2 in the atmosphere will be approximately 422.14 ppmv.

6. Discussion questions

1. Sketch the graph of the following functions.

(a) $f(x) = -5x + 12$ (b) $g(x) = x^2 - 7x + 10$

2. Find the intersection of the following two functions.

(a) $f(x) = 3x - 22, g(x) = -8x + 11$

(b) $f(x) = 2x^2 + x - 1, g(x) = x^2 + 6x + 5$

3. A call center purchased an operating system at a cost of \$600,000 in 2016 and has a scrap value of \$120,000 at the end of 8 years. Assuming the operating system depreciates in a linear fashion,

(a) Find the rate of depreciation.

(b) Find an expression giving the book value of the operating system at the end of t years.

(c) Find the book value of the operating system at the end of the fourth year.

4. AutoWipes, a manufacturer of windshield wipers, has a monthly fixed cost of \$48,000 and a production cost of \$8 for each windshield wipers manufactured. Each unit of windshield wiper sells for \$14.

(a) Find expressions for the total cost function, revenue function and profit functions.

(b) Find the break-even point.

(c) The company sells 10,000 wipers in December. Find the profit realised in December.

5. The supply function for a particular brand of torchlight is given by $p = 0.1x^2 + 0.5x + 15$ where x is the quantity supplied (in thousands) and p is the unit price in dollars.

(a) Sketch the graph of the supply function.

(b) What unit price will induce the supplier to make 5000 torchlights available in the market?

(c) What is the lowest price at which the supplier will make the torchlights available to the market?

6. The demand function for a type of electric shavers is given by $p = -x^2 - 2x + 100$ and the corresponding supply function is given by $p = 8x + 25$, where p is the unit price in dollars and x is in units of a thousand. Find the equilibrium quantity and price.

7. In 2009, the total global mobile data traffic was forecasted to be modelled by the function $f(t) = 0.021t^3 + 0.015t^2 + 0.12t + 0.06$, $0 \leq t \leq 10$ where $f(t)$ is measured in million terabytes per month and $t = 0$ corresponds to the year 2009.

(a) What was the total global mobile traffic in 2009?

(b) According to this model, what will be the total global mobile traffic in 2017?

7. Supplementary questions

1. Sketch the graph of the following functions.

(a) $f(x) = -3.5x - 2$ (b) $g(x) = 6x^2 - 11x - 10$

2. Find the intersection of the following two functions.

(a) $f(x) = -5x + 2, g(x) = -x^2 - 5x + 6$

(b) $f(x) = 0.2x^2 - 1.2x - 4, g(x) = -0.3x^2 + 0.7x + 8.2$

3. In 2016, the management of AAB batteries installed a machine in one of its factories at a cost of \$500,000. The machine is depreciated linearly over 20 years with a scrap value of \$20,000.

(a) Find an expression giving the book value of the machine in the *t* th year of use.
(b) Sketch the graph in part (a).
(c) Find the rate of depreciation.
(d) Find the book value of the machine in the eleventh year.

4. Nok Corporation manufactures mobile phone covers for a particular brand of mobile phone. Each cover sells for \$10 and the variable cost for producing each cover is 40% of its selling price. The monthly fixed costs are estimated to be \$30,000

(a) Find expressions for the total cost function, revenue function and profit functions.
(b) Find the break-even point.
(c) The CEO of the corporation wants the profit in January to reach \$12,000. Find the number of mobile phone covers that must be sold to achieve this target.

5. The weekly demand and supply functions for a certain type of tyre for trucks is given by

$$p = D(x) = 144 - x^2$$

$$p = S(x) = 48 + \frac{1}{2}x^2$$

where *p* is measured in dollars and *x* is measured in units of a thousand.

(a) Find the unit price that will have create a demand of 6000 tyres in the market.
(b) What is the maximum quantity demanded per week?
(c) Find the equilibrium quantity and price.

6. A study regarding a local seaside town predicts that the population of the town will increase according to the function

$$P(x) = 50,000 + 30x^{\frac{3}{2}} + 20x$$

for the next 5 years, where $P(x)$ denotes the population *x* months from now.

(a) What is the projected population 1 year from now?
(b) According to this model, will the population of the town exceed 70,000 within the next 5 years?

BUSINESS MATHEMATICS

SESSION 5

EXPONENTIAL AND LOGARITHM FUNCTIONS

At the end of the session, students should be able to:

1. Understand the relation between the exponential and logarithm functions.
2. Apply the laws of exponential and logarithm functions to solve equations.
3. Sketch the graphs of exponential and logarithm functions.
4. Solve problems related to exponential growth, exponential decay and logistic growth models.

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1. Introduction

In the previous session, we discussed the use of polynomial functions in mathematical modelling. This session will explore another two groups of functions which are used extensively in business and economics, namely the exponential and logarithm functions. We will first examine the basic notions of these functions followed by its application in growth and decay problems.

In particular, we will focus on the use of the constant, *e*, in these functions. The number *e* has been determined to be an irrational number, close to the value of 2.7182818. First introduced in the 1600s by John Napier in developing logarithms and expanded on by Leonard Euler in the 18^{th} century, the number *e* has enjoyed tremendous popularity in mathematics modelling, making its appearance in various unexpected contexts, such as compound interest, heating and cooling models, population growth and probability. Because of its widespread use, it is called the natural exponent.

In studying the use of *e*, mathematicians have realised that many natural quantities grow or decay in proportion to their size, e.g. population, financial savings. These quantities have been found to follow an exponential growth or decay model. At times, however, there is a limit to the amount that the quantity can achieve, e.g. a viral infection in an enclosed area. When this situation occurs, the quantity may follow a logistic growth model. Although there are many other applications in real life, we will only restrict our study to only these three models in this session.

2. Exponential functions

An exponential function with base *b* and exponent *x*, where *b* is a positive number, is defined by

$$f(x) = b^x, \quad (b > 0, b \neq 1)$$

The domain of *f* is the set of all real numbers.

2.1 Laws of exponents

Operations involving exponential functions use the laws of exponents stated in the first session. The laws are restated in Table 5.1 below.

Table 5.1: Laws of exponents

No.	Property	Example
1.	$b^x \cdot b^y = b^{x+y}$	$36^{\frac{1}{3}} \cdot 36^{\frac{1}{6}} = 36^{\frac{1}{3}+\frac{1}{6}} = 36^{\frac{1}{2}} = 6$
2.	$\dfrac{b^x}{b^y} = b^{x-y}$	$\dfrac{16^{\frac{7}{4}}}{16^{\frac{3}{2}}} = 16^{\frac{7}{4}-\frac{3}{2}} = 16^{\frac{1}{4}} = 2$
3.	$\left(b^x\right)^y = b^{xy}$	$\left(32^{\frac{1}{3}}\right)^{\frac{6}{5}} = 32^{\frac{1}{3}\cdot\frac{6}{5}} = 32^{\frac{2}{5}} = \left(32^{\frac{1}{5}}\right)^2 = 2^2 = 4$
4.	$\left(ab\right)^x = a^x b^x$	$\left(16 \cdot 49\right)^{\frac{1}{2}} = 16^{\frac{1}{2}} \cdot 49^{\frac{1}{2}} = 4 \cdot 7 = 28$
5.	$\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$	$\left(\dfrac{3^{\frac{1}{2}}}{5^{\frac{1}{3}}}\right)^6 = \dfrac{3^{\frac{1}{2}\cdot 6}}{5^{\frac{1}{3}\cdot 6}} = \dfrac{3^3}{5^2} = \dfrac{27}{25}$

2.2 Solving exponential equations

To solve basic exponential equations, we first observe whether both sides of the equation can be expressed in terms of the same base. If it is possible, we will use the property that

$$b^m = b^n \Leftrightarrow m = n$$

Example 5.1 Solve the equation $2^{3x-1} = \sqrt{2}$.

Solution

Converting both sides of the equation to exponent form, we have $2^{3x-1} = 2^{\frac{1}{2}}$.
Since the base on both sides are the same, we can equate the exponents.

$$3x - 1 = \frac{1}{2}$$

$$x = \frac{1}{2}$$

2.3 Graphing exponential functions

Before we can identify the properties of an exponential function, we will first inspect the graph of a specific case, namely $f(x) = 2^x$. The table below gives the values of $f(x)$ for different values of x.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

Note the following observations:

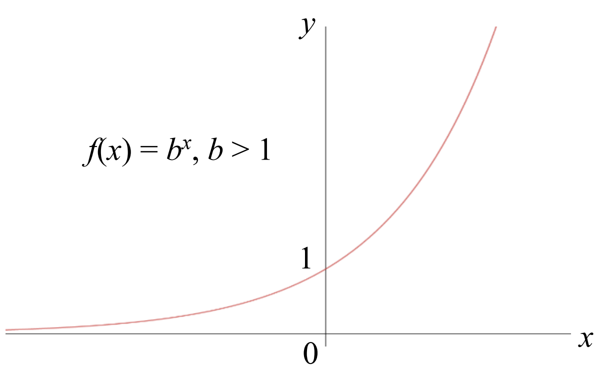
1) When x increases, 2^x increases at a faster rate than x and is not bounded by any value.
2) When x decreases, 2^x decreases and approaches zero. However, as there is no value of x for which $2^x = 0$, there is no x-intercept.
3) The y-intercept is 1 because $2^0 = 1$.

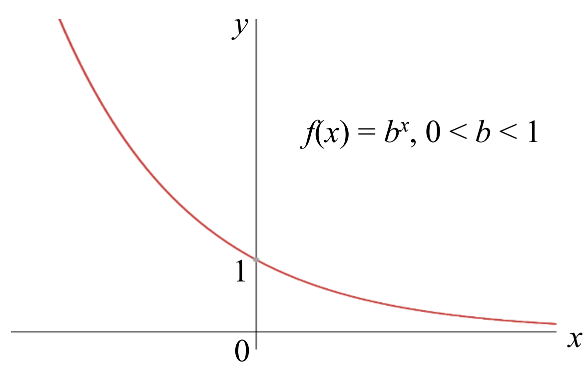
These properties can be generalised to other values of the base b, provided that $b > 1$. Do these properties hold when $0 < b < 1$? We inspect the case when $f(x) = (1/2)^x$ or equivalently, $f(x) = 2^{-x}$. A table of values is computed below.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

The pattern appears to be the reverse of $f(x) = 2^x$. When x increases, 2^{-x} decreases and approaches zero. When x decreases, 2^{-x} increases without bounds. There is no x-intercept and the y-intercept remains at 1. These properties can be extended to other exponential functions where $0 < b < 1$. The graphs for the two cases are shown below in Figure 5.1.

Figure 5.1: Graphs of the exponential function $f(x) = b^x$





The two examples above illustrate the following properties of exponential functions, $f(x) = b^x$.

1. The domain of f is the set of all real numbers. The range of f is $(0, \infty)$.
2. Since $b^0 = 1$, the y-intercept is always 1.
3. If $b > 1$, the graph rises from left to right. If $0 < b < 1$, the graph falls from left to right.

2.4 The base e

In mathematical modelling with exponential functions, the constant e is used widely and in many separate contexts. It is an irrational number and is defined by the expression

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m$$

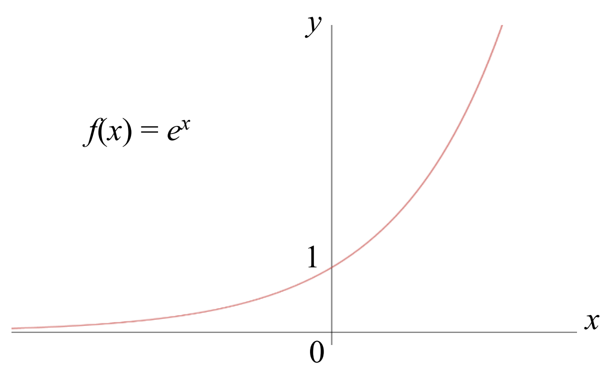
This means that as m gets larger and larger, the expression $\left(1+\frac{1}{m}\right)^m$ gets closer and closer to e, which is approximately 2.71828. The constant e is commonly known as the natural exponent.

Example 5.2 Compute the value of the function $f(x) = e^x$ for x = –3, –2, –1, 0, 1, 2, 3. Hence sketch the graph of $f(x) = e^x$.

Solution

x	–3	–2	–1	0	1	2	3
$f(x)$	0.05	0.14	0.37	1	2.72	7.39	20.09

Since $e \approx 2.71828 > 1$, the properties of the graph are similar to those discussed in the earlier section. The graph of $f(x) = e^x$ is sketched below.



3. Logarithm functions

Using the exponential function $b^y = x$, the logarithm of x to the base b, or $\log_b x$, is defined by

$$y = \log_b x \quad \text{if and only if} \quad b^y = x \qquad (b > 0, b \neq 1, x > 0)$$

Another way to consider a logarithm is that a logarithm is the exponent to which the base b must be raised to obtain the number x. Note that $\log_b x$ is a value and it is not equivalent to $\log_b$ times x. It is meaningless to write $\log_b$ by itself.

Recall that the exponential function $b^y = x$ is only defined for positive values of b and thus, only positive values of x are generated. Therefore, the logarithm $\log_b x$ is only defined for positive values of b and x.

Example 5.3 Express the following exponential equations in logarithm form.

(a) $10^3 = 1000$ (b) $64^{\frac{1}{2}} = 8$ (c) $15^1 = 15$

Solution

(a) $10^3 = 1000 \Leftrightarrow \log_{10} 1000 = 3$

(b) $64^{\frac{1}{2}} = 8 \Leftrightarrow \log_{64} 8 = \dfrac{1}{2}$

(c) $15^1 = 15 \Leftrightarrow \log_{15} 15 = 1$

Common and Natural logarithms

There are two numbers which are widely used as the base for logarithms. Logarithms with base 10 are called common logarithms and logarithms with base e are called natural logarithms. It is a standard practice to denote $\log_{10}$ as log and $\log_e$ as ln.

Example 5.4 Express the following logarithm equations in exponential form.

(a) $\log x = 3$ (b) $\ln x = 5$ (c) $\ln 2 = x$

Solution

(a) $\log x = 3 \Rightarrow \log_{10} x = 3 \Leftrightarrow x = 10^3$

(b) $\ln x = 5 \Rightarrow \log_e x = 5 \Leftrightarrow x = e^5$

(c) $\ln 2 = x \Rightarrow \log_e 2 = x \Leftrightarrow 2 = e^x$

3.1 Laws of logarithms

Operations involving logarithms are derived from the definition of a logarithm and corresponding laws of exponents. For example, if $\log_b m = p$ and $\log_b n = q$, then $p + q = \log_b m + \log_b n$.

However, from the definition of logarithm, we also have $b^p = m$ and $b^q = n$. Using the laws of exponents, $b^{p+q} = mn$. Applying the definition of logarithms again, we see that $p + q = \log_b mn$.

Equating the two expressions above, we have derived a formula to add two logarithms.

$$\log_b m + \log_b n = \log_b mn$$

Various other operations, such as subtraction and multiplying by a constant, on logarithms can be derived in a similar manner. These operations are known together as the laws of logarithms. Table 5.2 gives a summary of these laws.

Table 5.2: Laws of logarithms

No.	Property	Example
1.	$\log_b m + \log_b n = \log_b mn$	$\log 2 + \log 3 = \log(2\cdot 3) = \log 6$
2.	$\log_b m - \log_b n = \log_b \frac{m}{n}$	$\log 14 - \log 2 = \log\frac{14}{2} = \log 7$
3.	$n\log_b m = \log_b m^n$	$-2\log 3 = \log 3^{-2} = \log\frac{1}{9}$
4.	$\log_b 1 = 0$	$\log 1 = 0,\ \ln 1 = 0$
5.	$\log_b b = 1$	$\log_2 2 = 1,\ \ \ln e = 1$

When performing computations involving laws of logarithms, there are some common errors to avoid. In particular,

$\log_b \frac{m}{n}$ is **not equivalent** to $\frac{\log_b m}{\log_b n}$ and

$\log_b m^n$ is **not equivalent** to $(\log_b m)^n$.

Example 5.5 Expand and simplify the following expressions.

(a) $\log_4 16x$ (b) $\log_3 \frac{x^3+2}{3^x}$ (c) $\ln e^{2x}\sqrt{x}$

Solution

(a)
$$\begin{aligned}\log_4 16x &= \log_4 16 + \log_4 x && \text{(Law 1)}\\ &= \log_4 4^2 + \log_4 x\\ &= 2\log_4 4 + \log_4 x && \text{(Law 3)}\\ &= 2 + \log_4 x\end{aligned}$$

(b)
$$\begin{aligned}\log_3 \frac{x^3+2}{3^x} &= \log_3(x^3+2) - \log_3 3^x && \text{(Law 2)}\\ &= \log_3(x^3+2) - x\log_3 3 && \text{(Law 3)}\\ &= \log_3(x^3+2) - x && \text{(Law 5)}\end{aligned}$$

(c)
$$\begin{aligned}\ln e^{2x}\sqrt{x} &= \ln e^{2x} + \ln x^{\frac{1}{2}} && \text{(Law 1)}\\ &= 2x\ln e + \frac{1}{2}\ln x && \text{(Law 3)}\\ &= 2x + \frac{1}{2}\ln x && \text{(Law 5)}\end{aligned}$$

3.2 Solving exponential and logarithm equations

To solve an exponential equation, we first reduce the equation to the form $y = b^x$. If both sides of the equation can be expressed in terms of the same base, then we do so and equate the exponents. If this is not possible, we "take logarithms" on both sides of the equation to solve for x.

Example 5.6 Solve the following equations for x.

(a) $3^{x+1} = \frac{1}{27}$ (b) $3^{x+1} = 20$ (c) $2e^{x-1} + 14 = 150$

Solution.

(a)
$$\begin{aligned} 3^{x+1} &= \frac{1}{27} \\ 3^{x+1} &= 3^{-3} && \text{(Change to the same base)} \\ x+1 &= -3 && \text{(Equating exponents)} \\ x &= -4 \end{aligned}$$

(b) The number 20 cannot be expressed as an exponent with base 3.

$$\begin{aligned} \ln 3^{x+1} &= \ln 20 && \text{(Take natural logarithms on both sides)} \\ (x+1)\ln 3 &= \ln 20 && \text{(Using } n\log_b m = \log_b m^n\text{)} \\ x+1 &= \frac{\ln 20}{\ln 3} \\ &\approx 2.727 && \text{(Using your calculator)} \\ x &= 1.727 \end{aligned}$$

(c) First manipulate the equation to the form $y = b^x$.

$$\begin{aligned} 2e^{x-1} + 14 &= 150 \\ 2e^{x-1} &= 136 \\ e^{x-1} &= 68 \\ \ln e^{x-1} &= \ln 68 && \text{(Taking natural logarithm on both sides)} \\ (x-1)\ln e &= \ln 68 && \text{(Using } n\log_b m = \log_b m^n\text{)} \\ x-1 &= \ln 68 && \text{(Using } \log_b b = 1\text{)} \\ &\approx 4.2195 && \text{(Using your calculator)} \\ x &= 5.2195 \end{aligned}$$

To solve a logarithm equation, we first reduce it to the form $y = \log_b x$. The logarithm equation is usually transformed to its exponential equivalent, by using the definition of a logarithm, to solve for x.

Example 5.7 Solve the following equations for *x*.

(a) $\log_2 x = 6$ (b) $\log_x 49 = 2$ (c) $2\ln x = 11$

Solution

(a) $\log_2 x = 6$

$$x = 2^6 \quad \text{(Using the definition of logarithms)}$$
$$= 64$$

(b) $\log_x 49 = 2$

$$x^2 = 49$$
$$x = 7 \quad \text{(reject } -7 \text{ because } x > 0\text{)}$$

(c) We will first need to manipulate the equation to the form $y = \log_b x$.

$$2\ln x = 11$$
$$\ln x = 5.5$$
$$x = e^{5.5}$$
$$= 244.69$$

Example 5.8 Solve the following equations for *x*.

(a) $\log_2 x + \log_2 5 = 7$ (b) $2\ln x - \ln 3 = 2$ (c) $\log(x+1) - \log(x-1) = 1$

Solution

(a) $\log_2 x + \log_2 5 = 7$

$$\log_2 5x = 7 \quad (\log_b m + \log_b n = \log_b mn)$$
$$5x = 2^7 \quad \text{(Using the definition of logarithm)}$$
$$= 128$$
$$x = \frac{128}{5}$$

(b) $2\ln x - \ln 3 = 2$

$$\ln x^2 - \ln 3 = 2 \quad (n\log m = \log m^n)$$
$$\ln\frac{x^2}{3} = 2 \quad (\log_b m - \log_b n = \log_b \frac{m}{n})$$
$$\frac{x^2}{3} = e^2 \text{ (Using the definition of logarithm)}$$
$$x^2 = 3e^2$$
$$x = \sqrt{3}e$$
$$\approx 4.708$$

(c) $\log(x+1) - \log(x-1) = 1$

$$\log\frac{x+1}{x-1} = 1 \qquad (\log_b m - \log_b n = \log_b \frac{m}{n})$$

$$\frac{x+1}{x-1} = 10^1 \qquad \text{(Using the definition of logarithm)}$$

$$x+1 = 10(x-1)$$

$$= 10x - 10$$

$$9x = 11$$

$$x = \frac{11}{9}$$

3.3 Graphing logarithm functions

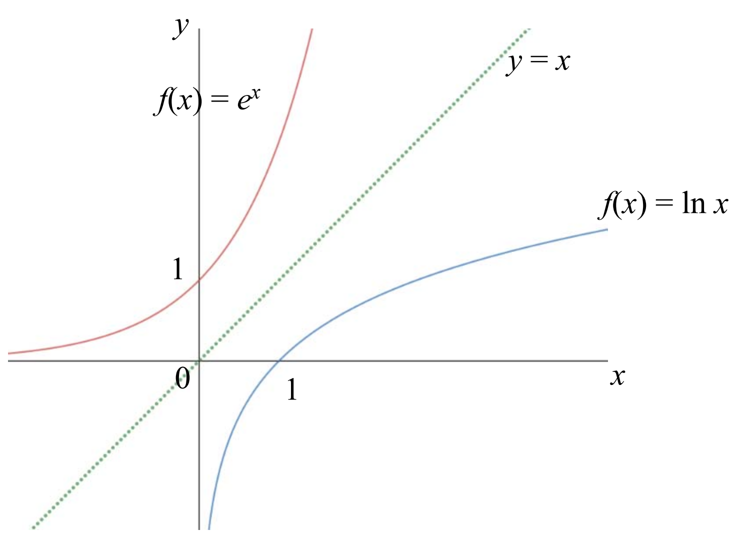
The properties of the logarithm function with base b, $f(x) = \log_b x$, are given below.

1. Since the logarithm function is defined for only $x > 0$, the domain of f is $(0, \infty)$. The range of f is $(-\infty, \infty)$.
2. As x can never be zero, there is no y-intercept.
3. Since $\log_b 1 = 0$, the x-intercept is always 1.
4. If $b > 1$, the graph rises from left to right. If $0 < b < 1$, the graph falls from left to right.
5. The graph of $f(x) = \log_b x$ is a mirror reflection of the graph $f(x) = b^x$ about the line $y = x$.

Example 5.9 Sketch the graph of $f(x) = e^x$ and $f(x) = \ln x$ on the same axes.

Solution

The graph of $f(x) = e^x$ is sketched as in Example 5.2. To sketch the graph of $f(x) = \ln x$, note that it is the mirror reflection of $f(x) = e^x$ about the line $y = x$.



3.4 Relation between exponential and logarithm functions

We have already seen that the graphs of the exponential function $f(x) = e^x$, and the logarithm function $f(x) = \ln x$, are mirror images of each other about the line $y = x$. This relationship is further described algebraically by the following.

$$e^{\ln x} = x$$

To see why this relation is true, suppose $e^y = x$. By the definition of logarithm, $y = \log_e x = \ln x$. Substituting this value for y into the original equation will give us $e^{\ln x} = x$.

Example 5.10 Solve the equation $10\ln x - 7 = 0$.

Solution

$$\begin{aligned} 10\ln x - 7 &= 0 \\ 10\ln x &= 7 \\ \ln x &= \frac{7}{10} = 0.7 \end{aligned}$$

Method 1: Using the definition of logarithms,

$$\begin{aligned} x &= e^{0.7} \\ &\approx 2.014 \end{aligned}$$

Method 2: Using the relation between exponential and logarithm functions,

$$\begin{aligned} e^{\ln x} &= e^{0.7} \\ &\approx 2.014 \end{aligned}$$

Since $e^{\ln x} = x$, $x \approx 2.014$.

4. Mathematical models with exponential functions

4.1 Exponential growth and decay model

Many real world quantities are closely related to the exponential function, in which they grow at a rate directly proportion to their amount at time t. Some examples would be the growth of bacteria in a culture or the interest earned in a savings account when compounded continuously. These quantities are said to exhibit exponential growth. Suppose $Q(t)$ represents a quantity at time t, then the mathematical model is given by

$$Q(t) = Q_0 e^{kt}$$

where Q_0 is the quantity that is initially present ($t = 0$) and the number k is a constant known as the growth constant.

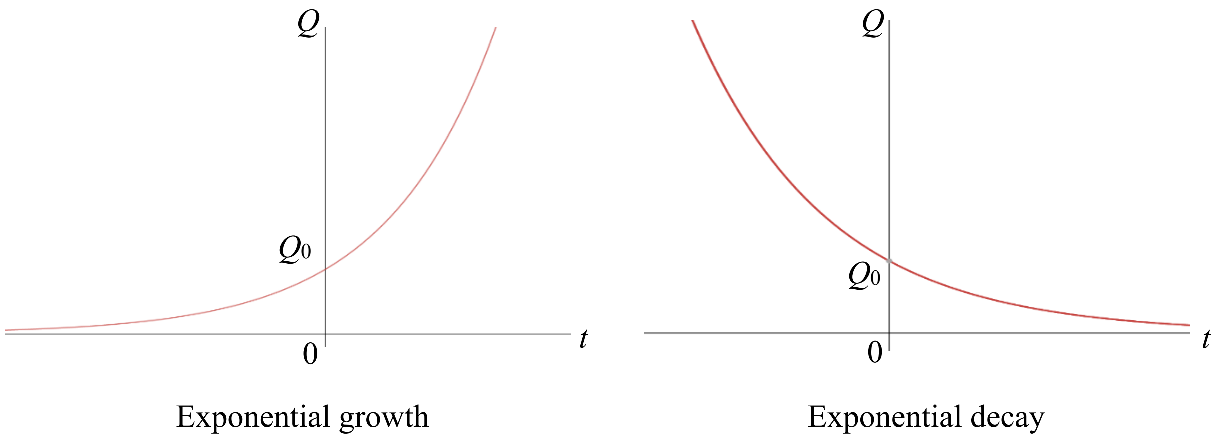
Similarly, if a quantity decreases at a rate directly proportional to their amount at time t, then the quantity is said to exhibit exponential decay. Suppose $Q(t)$ represents a quantity at time t, then the mathematical model for exponential decay is given by

$$Q(t) = Q_0 e^{-kt}$$

where Q_0 is the quantity that is initially present ($t = 0$) and the number k is a constant known as the decay constant.

The graphs modelling exponential growth and exponential decay are shown in Figure 5.2.

Figure 5.2: Graphs of exponential growth and exponential decay



Example 5.11 A multi-national company purchased a set of printing machines for \$500,000. After 3 years, the value of the machines had decreased to \$350,000. Assuming that the machine's value decreases exponentially, what will its value be after 7 years?

Solution
Since the value decreases exponentially, it follows the exponential decay function $Q(t) = Q_0e^{-kt}$, where $Q(t)$ is the value of the machine at time t, Q_0 is the initial value of the machine and k is the decay constant.

The initial value is \$500,000, i.e. $Q(0) = 500{,}000$.

$$Q_0 e^{-k(0)} = 500{,}000$$
$$Q_0 = 500{,}000 \qquad \text{(since } e^0 = 1\text{)}$$

To obtain the decay constant k, note that the value of the machine after 3 years is \$350,000, i.e. $Q(3) = 350{,}000$.

$$500{,}000e^{-k(3)} = 350{,}000$$
$$e^{-3k} = 0.7$$

$$-3k = \ln 0.7$$
$$k = 0.11889$$

Thus the exponential decay model is given by $Q(t) = 500{,}000e^{-0.11889t}$.

The value after 7 years is given by $t = 7$.

$$Q(7) = 500{,}000e^{-0.11889(7)}$$
$$= \$217{,}538.99$$

Example 5.12 A division of Western Mobile produces a certain type of mobile phone. The training and development section has determined that after completing a basic training program, a new employee will be able to assemble $Q(t) = 40 - 20e^{-0.5t}$ mobile phones per day, where t is the number of months after which the employee starts work on the assembly line.

(a) How many mobile phones can a new employee assemble per day after completing the basic training program?

(b) How many mobile phones can an employee with 6 months experience assemble per day?

(c) What is the greatest number of mobile phones an experienced employee can assemble per day?

Solution

(a) Immediately after completing the basic training program, $t = 0$.

$$Q(0) = 40 - 20e^{-0.5(0)} = 20$$

The number of mobile phones a new employee can assemble is 20.

(b) $Q(6) = 40 - 20e^{-0.5(6)} \approx 39$

An employee with 6 months experience can assemble 39 mobile phones per day.

(c) As t gets larger and larger, $e^{-0.5t}$ gets closer and closer to 0. This implies that $40 - 20e^{-0.5t}$ gets closer and closer to 40. Hence greatest number of mobile phones = 40.

4.2 Logistic growth model

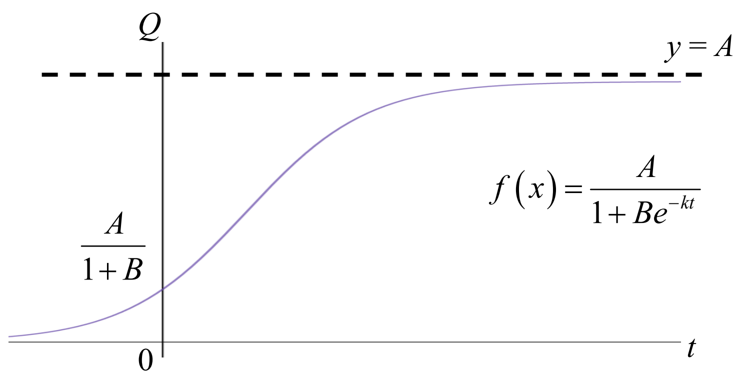
It is possible that a quantity exhibits exponential growth at the beginning stages and reaches a "saturation" stage as time increases. For example, a small population might increase at a fast rate in the early years but as time goes by, scarcity of resources would cause the population to stabilise at a certain number. Quantities which have these characteristic are said to follow a logistic growth model. The logistic growth function that models this is given by

$$Q(t) = \frac{A}{1 + Be^{-kt}}$$

where $Q(t)$ represents the quantity at time t and the values A, B and k are positive constants.

Figure 5.3 shows the graph of a logistic growth function. Observe that the curve is shaped like the letter S. The growth rate increases at first but decreases quite rapidly as t increases. As the value of t gets larger, the quantity approaches the number A but never exceeds A.

Figure 5.3: A logistic growth function



Example 5.13 The number of students who contracted influenza in a private school after t days during an epidemic is approximated by the model

$$Q(t) = \frac{6000}{1+1500e^{-kt}}.$$

After 6 days, 50 students had contracted flu. How many students will contract flu after 10 days?

Solution

Note that 50 students contracted flu after 6 days, i.e. $Q(6) = 50$.

$$\begin{aligned} \frac{6000}{1+1500e^{-6k}} &= 50 \\ 6000 &= 50\left(1+1500e^{-6k}\right) \\ 120 &= 1+1500e^{-6k} \\ e^{-6k} &= 0.07933 \\ -6k &= \ln 0.07933 \\ k &= 0.42235 \end{aligned}$$

Hence the logistic growth function is $Q(t) = \dfrac{6000}{1+1500e^{-0.42235t}}$.

After 10 days, the number of students who contracted flu is given by

$$\begin{aligned} Q(10) &= \frac{6000}{1+1500e^{-0.42235(10)}} \\ &= 261.199 \\ &\approx 261 \end{aligned}$$

5. Discussion questions

1. Solve the following equations.

(a) $(5.7)^{2x-3} = (5.7)^{x+1}$ (b) $e^{0.5x+1} = e^{5x-8}$

(c) $3^{2x} = \left(\frac{1}{27}\right)^{5-3x}$ (d) $3^{2x} - 12(3^x) + 27 = 0$

2. Sketch the following functions on the same axes: $f(x) = e^{2x}$, $g(x) = e^{0.5x}$.

3. Expand and simplify the following expressions.

(a) $\log x(x+1)^6$ (b) $\ln \frac{e^x}{1+e^x}$

4. Solve the following equations.

(a) $\log_5 x = 2$ (b) $\log_x 100 = 2$

5. Solve the following equations.

(a) $e^{0.2x} = 5$ (b) $10 + 2e^{3x} = 50$

(c) $\ln(2x+1) = 5$ (d) $2\ln x + \ln 6 = 9$

6. In a study conducted by an aviation authority, air travel will rise in the next 10 years given according to the exponential model $f(t) = 666e^{0.0413t}$, $0 \leq t \leq 10$, where $f(t)$ is the number of passengers (in millions) and t is the number of years from now.

(a) How many air passengers are there currently?
(b) How many air passengers will there be in 8 years time?

7. The temperature of a cup of coffee at time t (in minutes) is given by the function

$$T(t) = 25 + ce^{-kt}$$

where c and k are constants. Initially the temperature was 80ºC. Three minutes later it was 55ºC. When will the temperature of the coffee be 30ºC?

8. Data collected in an experiment concerning the growth of a population of butterflies with a limited supply of food showed that the population growth can be approximated by

$$N(t) = \frac{400}{1+39e^{-0.16t}}$$

Where t is the number of days since the beginning of the experiment.

(a) What was the initial population of butterflies in the experiment?
(b) What was the population of butterflies on the 15th day?

6. Supplementary questions

1. Solve the following equations.

(a) $(-1.6)^{x+13} = (-1.6)^{-x+1}$ (b) $e^{2x-1} = e^{3x+16}$

(c) $16^{x} = 32^{1-x}$ (d) $2^{2x} - 4\left(2^{x}\right) + 4 = 0$

2. Sketch the following functions on the same axes: $f(x) = e^{-2x}$, $g(x) = e^{-0.5x}$.

3. Expand and simplify the following expressions.

(a) $\log \dfrac{\sqrt{x-1}}{1000}$ (b) $\ln \dfrac{e^{2x}}{\sqrt{x}}$

4. Solve the following equations.

(a) $\log_7 x = 3$ (b) $\log_x 729 = 3$

5. Solve the following equations.

(a) $2e^{0.4x-1} = 15$ (b) $\dfrac{200}{1+5e^{-0.4x}} = 90$

(c) $\ln(3x-11) = 4$ (d) $2\ln x - 3\ln 2 = 1$

6. The monthly demand for a new brand of HD televisions t months after introducing it on the market is given by the demand function $D(t) = 4000 - 3000\, e^{-0.06t}$.

(a) What was the demand after 1 year?
(b) After how many months will the demand reach 3500?
(c) At what level is the demand expected to stabilise?

7. 200 executives from a multinational company attended the annual dinner where the chief executive officer announced a new company policy. Two hours later, 400 executives had heard about the new policy. If the number of executives who heard of the new policy t hours later is given by the function

$$f(t) = \frac{3000}{1+Be^{-kt}},$$

how many executives will have heard about the new policy after 4 hours?

BUSINESS MATHEMATICS

SESSION 6

MATHEMATICS OF FINANCE

At the end of the session, students should be able to:

1. Understand and apply the concepts of simple interest, compound interest, continuous compounding of interest and effective rate of interest.
2. Understand and apply the concept of future and present value of annuities, amortization of loans and sinking funds.

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1. Introduction

Managing your finances requires much planning and thought. You will need to meet all your daily needs and at the same time consider your long term goals such as buying a house or investing for retirement. You might also be tasked to handle a companies' finances. This session will provide information on doing so by examining the concepts of simple interest, compound interest and annuities. In particular, we will answer questions involving the amortization of loans and the setting up of sinking funds using the formulas for annuities.

Interest is the cost of using someone else's money. When you borrow money, you will need to repay what you borrowed and compensate the lender for the risk of lending to you. When you have extra money, you can also lend money out yourself or deposit in a bank and let the bank offer loans to other customers. In either case, you will also expect to earn back more than what you lent out. So how much would you pay or earn in interest? In many cases, it depends on both the interest rate and whether the interest is computed as simple or compound interest.

On the other hand, an annuity is a contract between a person and a company in which there is a sequence of payments made at regular time intervals. For example, you might make a series of payments into a retirement account when you are in the workforce. In return, you will obtain regular payouts when you retire. Nowadays, annuities can come mainly in three varieties: a fixed annuity that pays out a guaranteed amount based on your account balance, a variable annuity that gives returns based on the performance of the investments in your account and an indexed annuity that is a combination of both. In this chapter, we will restrict our study to only fixed annuities and their applications.

2. Simple and Compound Interest

A natural application of functions in business and finance is found in the computations of simple and compound interest.

2.1 Simple interest

Interest that is computed on the original principal only is known as simple interest. Suppose P denotes the principal amount and r the interest rate per year. The total interest I, after t years can be computed by

$$I = P\,r\,t$$

The accumulated amount A, i.e. the sum of the principal and interest, after t years is given by

$$A = P + I = P + P\,r\,t = P\left(1 + r\,t\right)$$

Example 6.1 A bank pays simple interest at the rate of 6% per year for their savings account. If a person makes a deposit of \$1000 and makes no withdrawal for 3 years, what is the interest earned in that period of time? What is the total amount of deposit at the end of 3 years?

Solution

$$\begin{aligned}\text{Interest earned} &= P\,r\,t\\ &= 1000\left(0.06\right)\left(3\right)\\ &= 180 \quad or \quad \$180\end{aligned}$$

$$\begin{aligned}\text{Total amount} &= P\left(1 + r\,t\right)\\ &= 1000\left[1 + \left(0.06\right)\left(3\right)\right].\\ &= 1180 \quad or \quad \$1180\end{aligned}$$

Year	Amount	Interest	Total amount
1	\$1000	\$60	\$1060
2	\$1060	\$60	\$1120
3	\$1120	\$60	\$1180

2.2 Compound interest

Interest that is computed on the original principal and any accumulated interest is known as compound interest. Suppose P denotes the principal amount and r the nominal interest rate per year. The accumulated amount at the end of t years can be computed by the following steps:

Accumulated amount after 1st year = $P(1 + rt) = P(1 + r)$

The interest in the 2nd year is computed based on the accumulated amount in the 1st year, i.e. the new principal amount is now $P(1 + r)$.
Hence accumulated amount after 2nd year = $[P(1 + r)](1 + r) = P(1 + r)^2$

The interest in the 3rd year is computed based on the accumulated amount in the 2nd year, i.e. the new principal amount is now $P(1 + r)^2$.
Hence accumulated amount after 3rd year = $[P(1 + r)^2](1 + r) = P(1 + r)^3$

Continuing this pattern, it would appear that the accumulated amount A after t years is given by the formula $A = P(1 + r)^t$.

However, the pattern above was derived under the assumption that interest is paid out only once a year. In this case, we say that interest is compounded once a year. In practice, interest is usually paid out several times in a year, i.e. it is compounded more than once a year. We call the interval of time between successive interest calculations the conversion period. The number of conversion periods per year (or the number of times interest is calculated in a year) is denoted by *m*.

If interest at a nominal rate of *r* per year is compounded *m* times a year, then the simple interest rate per conversion period can be re-calculated by dividing the annual interest rate with the number of conversion periods per year, i.e. $i = r / m$. If the interest is compounded for *t* years, then the total number of periods can be re-calculated by multiplying the number of conversion periods per year with the number of years, i.e. $n = mt$. The formula for the accumulated amount for compound interest can now be is modified to:

$$A = P(1+i)^n$$

where A = accumulated amount
P = principal amount
i = interest rate per period
n = total number of periods

The variables *i* and *n* are computed by the following.

$$i = \frac{r}{m},\ n = mt$$

where r = nominal interest rate per year
m = number of conversion periods per year
t = number of years

Example 6.2 The nominal interest rate per year on a savings account at a bank is 6%. If a person makes a deposit of \$1000, what is the accumulated amount after 3 years if the interest is compounded (a) annually, (b) semi-annually, (c) quarterly, (d) monthly and (e) daily?

Solution
In this question, $P = 1000$, $r = 0.06$, $t = 3$.

(a) $m = 1$ so $i = \frac{0.06}{1}, n = 1(3)$

$$A = 1000(1+0.06)^3$$
$$= 1191.02 \quad or \quad \$1191.02$$

Period	Amount	Interest	Total amount
1	\$1000	\$60	\$1060
2	\$1060	\$63.60	\$1123.60
3	\$1123.60	\$67.42	\$1191.02

(b) $m=2$ so $i=\dfrac{0.06}{2}, n=2(3)$

$$A=1000\left(1+\frac{0.06}{2}\right)^{2(3)}$$
$$=1194.05 \quad or \quad \$1194.05$$

Period	Amount	Interest	Total amount
1	$1000	$30	$1030
2	$1030	$30.90	$1060.90
3	$1060.90	$31.827	$1092.727
4	$1092.727	$32.782	$1125.509
5	$1125.509	$33.765	$1159.274
6	$1159.274	$34.778	$1194.052

(c) $m=4$ so $i=\dfrac{0.06}{4}, n=4(3)$

$$A=1000\left(1+\frac{0.06}{4}\right)^{4(3)}$$
$$=1195.62 \quad or \quad \$1195.62$$

Period	Amount	Interest	Total amount
1	$1000	$15	$1015
2	$1015	$15.225	$1030.225
…	…	…	…
12	$1177.949	$17.669	$1195.618

(d) $m=12$ so $i=\dfrac{0.06}{12}, n=12(3)$

$$A=1000\left(1+\frac{0.06}{12}\right)^{12(3)}$$
$$=1196.68 \quad or \quad \$1196.68$$

Period	Amount	Interest	Total amount
1	$1000	$5	$1005
2	$1005	$5.025	$1010.025
…	…	…	…
36	$1190.727	$5.954	$1196.681

(e) $m=365$ so $i=\dfrac{0.06}{365}, n=365(3)$

$$A=1000\left(1+\frac{0.06}{365}\right)^{365(3)}$$
$$=1197.20 \quad or \quad \$1197.20$$

Period	Amount	Interest	Total amount
1	$1000	$0.164	$1000.164
2	$1000.164	$0.164	$1000.328
…	…	…	…
365	$1197.003	$0.197	$1197.200

2.3 Continuous compounding of interest

As interest is computed more and more frequently, the larger the accumulated amount would be (See Example 6.2). A question that naturally arises would be: Is there a limit to the accumulated amount when it is compounded more and more frequently or does it keep growing without bounds?

To answer that question, we would need to re-examine the compound interest formula:

$$A=P(1+i)^n=P\left(1+\frac{r}{m}\right)^{mt}$$

Let $u = \frac{m}{r}$. Then the formula can be rewritten as

$$A = P\left(1 + \frac{1}{u}\right)^{urt} = P\left[\left(1 + \frac{1}{u}\right)^{u}\right]^{rt}.$$

As the number of conversion periods per year *m*, gets larger and larger, the value of *u* also gets larger and larger. We are interested to know what happens when *u* tends to infinity. From the previous chapter, we know that

$$\lim_{u \to \infty}\left(1 + \frac{1}{u}\right)^{u} = e \ (\approx 2.71828).$$

Thus as the number of conversion periods increases without bounds, the accumulated amount *A* approaches $P(e)^{rt}$. When this happens, we say that interest is compounded continuously. Thus, the formula for the accumulated amount, when interest is compounded continuously, is given by:

$$A = Pe^{rt}$$

Example 6.3 The annual interest rate on a savings account at a bank is 6%. If a person makes a deposit of \$1000, what is the accumulated amount after 3 years if the interest is compounded (a) daily and (b) continuously?

Solution
In this question, $P = 1000$, $r = 0.06$, $t = 3$.

(a) $m = 365$ so $i = \frac{0.06}{365}, n = 365(3)$

$$\begin{aligned} A &= 1000\left(1 + \frac{0.06}{365}\right)^{365(3)} \\ &= 1197.20 \quad or \quad \$1197.20 \end{aligned}$$

(b) Interest is compounded continuously so

$$\begin{aligned} A &= 1000e^{0.06(3)} \\ &= 1197.22 \quad or \quad \$1197.22 \end{aligned}.$$

2.4 Effective rate of interest

From the examples above, we see that the actual interest earned on an investment depends on the number of conversion periods in a year, or the frequency in which interest is compounded. As the nominal interest rate does not reflect the actual rate, we need to find a common basis so that we can easily compare interest rates. One method of doing so is to compute the effective rate. The effective rate is the simple interest rate that would produce the same accumulated amount in 1 year as the nominal rate compounded *m* times a year. It is also known as the annual percentage yield.

To derive a formula for effective rate, suppose that P denotes the principal amount. Note that the accumulated amount after 1 year at a simple interest rate of r_{eff} is given by $A = P(1 + r_{eff})$. Likewise, the accumulated amount after 1 year at an interest rate of r per year compounded m times is given by $A = P\left(1+\frac{r}{m}\right)^m$. Equating the two expressions and solving for r_{eff}, we thus obtain

$$r_{eff} = \left(1+\frac{r}{m}\right)^m - 1$$

Example 6.4 Find the effective rate of interest corresponding to a nominal rate of 6% per year compounded (a) annually, (b) semi-annually, (c) quarterly, (c) monthly, (d) daily?

Solution
In this question, $r = 0.06$.

(a) $m = 1$ so $r_{eff} = \left(1+\frac{0.06}{1}\right)^1 - 1 = 0.06 \quad or \quad 6\%$

(b) $m = 2$ so $r_{eff} = \left(1+\frac{0.06}{2}\right)^2 - 1 = 0.0609 \quad or \quad 6.09\%$

(c) $m = 4$ so $r_{eff} = \left(1+\frac{0.06}{4}\right)^4 - 1 = 0.06136 \quad or \quad 6.136\%$

(d) $m = 12$ so $r_{eff} = \left(1+\frac{0.06}{12}\right)^{12} - 1 = 0.06168 \quad or \quad 6.168\%$

(e) $m = 365$ so $r_{eff} = \left(1+\frac{0.06}{365}\right)^{365} - 1 = 0.06183 \quad or \quad 6.183\%$

2.5 Present value

Sometimes, a person might want to determine how much money to invest now, at a fixed rate of interest, so that he will realize a certain sum at a future date. This can easily be calculated using the compound interest formula in Section 2.2 by expressing P in terms of A. The principal P is often called the present value whereas the accumulated value A is often called the future value as it is realized in the future. Rearranging the formula gives us

$$P = A(1+i)^{-n}$$

where A = future value
P = present value
i = interest rate per period
n = total number of periods

For the case of continuous compounding of interest, the present value is obtained by rearranging the formula in Section 2.3. We will get

$$P = Ae^{-rt}$$

Example 6.5 How much money should be deposited in a savings account paying interest at the rate of 8% per year compounded monthly, so that at the end of 3 years, the accumulated value will be $10,000?

Solution

In this question, A = 10000, r = 0.08, t = 3, m = 12. Therefore $i = \frac{0.08}{12}$, n = 12(3).

$$\begin{aligned} P &= 10000\left(1+\frac{0.08}{12}\right)^{-12(3)} \\ &= 7872.55 \quad or \quad \$7872.55 \end{aligned}$$

Example 6.6 (Using logarithms to solve problems in finance)
How long will it take for $5,000 to accumulate to an amount of $10,000 if the investment earns an interest rate of 5% per year compounded quarterly?

Solution

In this question, P = 5000, A = 10000, r = 0.05, m = 4. Therefore $i = \frac{0.05}{4}$, $n = 4t$.

Using the formula for compound interest,

$$\begin{aligned} 10000 &= 5000\left(1+\frac{0.05}{4}\right)^{4t} \\ (1.0125)^{4t} &= \frac{10000}{5000} \\ &= 2 \end{aligned}$$

Taking natural logarithms on both sides of the equation,

$$\begin{aligned} \ln(1.0125)^{4t} &= \ln 2 \\ 4t \ln 1.0125 &= \ln 2 \\ 4t &= \frac{\ln 2}{\ln 1.0125} \\ &\approx 55.7976 \\ t &\approx 13.95 \end{aligned}$$

It will take approximately 13.95 years for an investment of $5,000 to accumulate to $10,000.

3. Annuities

An annuity is a sequence of payments made at regular time intervals. There are many different types of annuities. For example, an annuity could be an investment account with a bank or an insurance company. It could be an investment in stocks, bonds or mutual funds. An annuity can also be used to provide long term regular payments to individuals.

The time period in which these payments are made is called the term of the annuity. An ordinary annuity is an annuity in which payments are made at the end of each payment period and a simple annuity is one in which the payment period coincides with the interest conversion period. In this section, we will only consider ordinary annuities that are simple with a fixed interest rate for each compounding period.

3.1 Future value

To find a formula for the accumulated amount of an annuity, consider an example when \$1000 is paid into an account at the end of each month over the period of 12 months. Furthermore, the account earns interest at a rate of 12% per year compounded monthly.

The first deposit of \$1000 would earn interest over a period of 11 months. The accumulated value at the end of the period would be

$$A_1 = 1000\left(1+\frac{0.12}{12}\right)^{11} = 1000(1+0.01)^{11}.$$

The second deposit of \$1000 would earn interest over a period of 10 months. The accumulated value at the end of the period is $A_2 = 1000(1 + 0.01)^{10}$.

The pattern continues for the third, fourth, fifth and so on till the last deposit. The final deposit would earn no interest. The payments and their accumulated amounts are summarized in Table 6.1 below.

Table 6.1: Accumulated amounts of an annuity

Month	Payment	Accumulated amount at the end of 12 months
1	\$1000	$1000(1+0.01)^{11}$
2	\$1000	$1000(1+0.01)^{10}$
3	\$1000	$1000(1+0.01)^{9}$
…	…	…
11	\$1000	$1000(1+0.01)^{1}$
12	\$1000	1000

We will want to know the accumulated amount of the annuity, S. This can be computed by adding all the terms in the table, i.e.

$$S = 1000 + 1000(1+0.01)^1 + 1000(1+0.01)^2 + ... + 1000(1+0.01)^{11}$$

Observe that the terms are the first 12 terms of a geometric progression with first term 1000 and common ratio (1 + 0.01). From Chapter 2, this sum can be computed by

$$S = 1000\left[\frac{(1+0.01)^{12}-1}{0.01}\right] \approx \$12,682.50$$

Now suppose that a sum of $\$R$ is paid into an account at the end of each period for n periods and that the account earns interest at the rate of i per period. Proceeding in the same manner above, the accumulated amount or future value S of an annuity can be computed to be

$$S = R\left[\frac{(1+i)^n-1}{i}\right]$$

where S = future value of the annuity,
R = payment per period of the annuity,
i = interest rate per period,
n = total number of periods.

Example 6.7 A person deposits \$200 at the end of every month into a savings account. If the savings account pays interest at a rate of 8% per year compounded monthly, how much money would he have accumulated at the end of 10 years?

Solution

In this question, $R = 200$, $r = 0.08$, $t = 10$, $m = 12$. Therefore $i = \frac{0.08}{12}$, $n = 10(12)$. Thus,

$$S = 200\left[\frac{\left(1+\frac{0.08}{12}\right)^{10\times12}-1}{\frac{0.08}{12}}\right] = 36589.21 \quad or \quad \$36,589.21$$

3.2 Present value

In some cases, we might want to find the current value of a sequence of equal periodic payments that will be made. After each payment is made, the new balance continues to earn interest. This current value is known as the present value of the annuity.

To find a formula for the present value of an annuity, suppose that an amount P is invested now and earns interest at the rate of i per period. At the end of n periods, it will have an accumulated value of $P(1 + i)^n$. But this value must be equal to the future value of the annuity in the previous section. Equating the two expressions and solving for P, we obtain the following formula:

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

where P = present value of the annuity,
R = payment per period of the annuity,
i = interest rate per period,
n = total number of periods.

Example 6.8 Find the present value of an ordinary annuity consisting of 24 monthly payments of \$50 each and earning interest at 10% per year compounded monthly.

Solution

In this question, $R = 50$, $r = 0.1$, $t = 2$, $m = 12$. Therefore $i = \dfrac{0.1}{12}$, $n = 2(12) = 24$.

$$P = 50\left[\frac{1-\left(1+\dfrac{0.1}{12}\right)^{-24}}{\dfrac{0.1}{12}}\right] = 1083.54 \quad or \quad \$1,083.54$$

Example 6.9 After making a down payment of \$200 for a refrigerator, John paid \$40 per month for 2 years with interest charged at 6% per year compounded monthly on the unpaid balance. What was the original cost of the refrigerator?

Solution

The loan taken by John is given by the present value of the annuity. Here, $R = 40$, $r = 0.06$, $t = 2$, $m = 12$. Therefore $i = \dfrac{0.06}{12}$, $n = 2(12)$.

$$P = 40\left[\frac{1-\left(1+\dfrac{0.06}{12}\right)^{-24}}{\dfrac{0.06}{12}}\right] = 902.51 \quad or \quad \$902.51$$

Thus, original cost of refrigerator = \$902.51 + \$200 = \$1,102.51

3.3 Amortization of loans

The annuity formulas used in the previous sections may be used to solve problems involving the amortization of loans. For example, a family might take a housing loan from a bank that charges interest at a fixed rate on the unpaid amount of the debt. They would want to know the size of each monthly instalment that they would have to pay so that the loan would be amortized by the end of its term. As the unpaid amount of debt is equivalent to the present value of an annuity, we apply that particular formula and express R in terms of P. The following formula is obtained.

$$R = \frac{Pi}{1-(1+i)^{-n}}$$

where R = periodic payment,
P = loan amount,
i = interest rate per period,
n = total number of periods.

Example 6.10 A loan of $20,000 is to be repaid through equal instalments made at the end of each year. An interest rate of 5% per year is charged on the unpaid balance and interest calculations are made at the end of each year. Calculate the size of each instalment so that the loan is amortized at the end of 4 years.

Solution
In this question, we are interested to find the amount of each instalment, i.e. the value of R.
$P = 20000$, $r = 0.05$, $t = 4$, $m = 1$. Therefore $i = \frac{0.05}{1}$, $n = 1(4)$.

$$R = \frac{20000\left(\frac{0.05}{1}\right)}{1-\left(1+\frac{0.05}{1}\right)^{-4\times 1}} = 5640.24 \quad or \quad \$5,640.24$$

The amortization schedule is reflected below to illustrate the interest payments.

End of year	Interest charged	Repayment made	Payment toward principal	Outstanding principal
0	-	-	-	$20,000.00
1	$1,000	$5,640.24	$4,640.24	$15,359.76
2	$767.99	$5,640.24	$4,872.25	$10,487.51
3	$524.38	$5,640.24	$5,115.86	$5,371.65
4	$268.59	$5,640.24	$5,371.65	-

Example 6.11 Jane secures a housing loan of $500,000 for a term of 30 years from a bank to finance her purchase of her home. The bank charges interest at a rate of 4% per year compounded monthly on the unpaid balance.
(a) What is her monthly payment?
(b) After repaying the loan for 5 years, Jane decides refinance her loan with another bank for the remaining term. The other bank charges an interest rate of 3.8% per year compounded monthly What is the outstanding principal owed by Jane after 5 years? What is her new monthly payment?

Solution

(a) The monthly payment is computed from the original loan amount and term.

P = 500,000, r = 0.04, t = 30, m = 12. Therefore $i=\dfrac{0.04}{12}$, n = 12(30).

$$R=\frac{500000\left(\frac{0.04}{12}\right)}{1-\left(1+\frac{0.04}{12}\right)^{-30\times 12}}=2387.08 \quad or \quad \$2,387.08$$

(b) The outstanding principal is given by the present value of her remaining payments.

R = 2,387.08, r = 0.04, t = 25, m = 12. Therefore $i=\dfrac{0.04}{12}$, n = 12(25).

$$P=2387.08\left[\frac{1-\left(1+\frac{0.04}{12}\right)^{-300}}{\frac{0.04}{12}}\right]=452238.23 \quad or \quad \$452,238.23$$

The new monthly payment is computed from the outstanding loan and remaining term.

Thus, P = 452,238.23, r = 0.038, t = 25, m = 12. Therefore $i=\dfrac{0.038}{12}$, n = 12(25).

$$R=\frac{452238.23\left(\frac{0.038}{12}\right)}{1-\left(1+\frac{0.038}{12}\right)^{-300}}=2337.42 \quad or \quad \$2,337.42$$

3.4 Sinking funds

Another important application of the annuity formulas is in the use of sinking funds. A sinking fund is an account that is set up to accumulate funds that will be used for a specific purpose in the future. For example, a town council may set up a sinking fund to build up funds for the purposes of setting up a children's playground. The council would want to know the size of each payment that goes into the sinking fund so that sufficient funds could be accumulated by a certain period of time. As the amount to be set up is equivalent to the future value of an annuity, we apply that particular formula and express R in terms of S. The following formula is obtained.

$$R=\frac{Si}{(1+i)^n-1}$$

where R = periodic payment,
S = sum to be accumulated,
i = interest rate per period,
n = total number of periods.

Example 6.12 To renovate its store front, the management of Faith's florist has decided to set up a sinking fund for this purpose in 3 years' time. It is expected that the renovation will cost \$50,000. If the fund earns 12% interest per year compounded semi-annually, determine the size of each semi-annual payment the company should pay into the fund.

Solution
We want to find the size of each payment, i.e. the value of R. In this question, $S = 50000$, $r = 0.12$, $t = 3$, $m = 2$. Therefore $i = \dfrac{0.12}{2}$, $n = 2(3)$.

$$R = \frac{50000\left(\dfrac{0.12}{2}\right)}{\left(1+\dfrac{0.12}{2}\right)^{3\times 2} - 1} = 7168.13 \quad or \quad \$7,168.13$$

The sinking fund schedule is reflected below to illustrate the interest payments.

End of period	Deposit made	Interest earned	Addition to fund	Accumulated amount in fund
1	\$7,168.13	0	\$7,168.13	\$7,168.13
2	\$7,168.13	\$430.09	\$7,598.22	\$14,766.35
3	\$7,168.13	\$885.98	\$8,054.11	\$22,820.46
4	\$7,168.13	\$1,369.23	\$8,537.36	\$31,357.82
5	\$7,168.13	\$1,881.47	\$9,049.60	\$40,407.42
6	\$7,168.13	\$2,424.44	\$9,592.57	\$50,000.00

4. Discussion questions

1(a) Find the simple interest on a \$2000 investment made for 3 years at an interest rate of 5% per year.

(b) Find the accumulated amount after 6 years if \$7500 is invested at an interest rate of 7% per year compounded (i) annually, (ii) monthly, (iii) continuously.

2. Find the effective rate of interest corresponding to a nominal rate of 9% per year compounded (i) quarterly, (ii) monthly.

3. Find the present value of \$50,000 due in 6 years at an interest rate of 8% per year compounded quarterly.

4(a) To help finance the purchase of a new television set, Amy applied for a short term loan in the amount of \$3,000 for a term of 6 months. If the bank charges simple interest at the rate of 12% per year, how much will Amy owe the bank at the end of 6 months?

(b) Barry was the beneficiary of a trust fund that was established for him 20 years ago. The original amount of the trust fund was \$15,000. How much will he receive now if the fund earns interest at the rate of 8% per year compounded monthly?

5(a) Find the future value of an ordinary annuity with a payment of \$1,500/month for 10 years at an interest rate of 5%/year compounded monthly.

(b) Find the present value of an ordinary annuity with a payment of \$2,000/quarter for 8 years at an interest rate of 10%/year compounded quarterly.

6(a) Find the periodic payment required to amortize a loan of \$100,000 over 20 years with interest charged at the rate of 8% per year compounded monthly,

(b) Find the periodic payment required to accumulate a sum of \$100,000 over 20 years with interest charged at the rate of 8% per year compounded monthly.

7. Sharon wants to create a retirement account for herself. Her financial advisor estimates that she would need to accumulate a total amount of \$1,000,000 in order to achieve a comfortable retirement.

(a) Determine the monthly deposit she has to make to a savings account that earns interest at 5% per year compounded monthly in order to retire comfortably in 30 years.

(b) If Sharon wishes to retire in 20 years, her advisor suggests depositing \$2,400 each month into a savings account that earns interest at 5.5% per year compounded monthly. Calculate the total amount that she will have accumulated in 20 years in this scenario.

5. Supplementary questions

1(a) Michael took a loan of $5000 from his cousin for a period of 6 months. His cousin charges him a simple interest rate of 6% per year. How much would he have to pay at the end of the term?

(b) Ten years ago, Christopher invested $20,000 in a retirement fund that grew at the rate of 3% per year compounded quarterly. How much is his fund worth today?

2. A bank charges an interest rate of 6% per year compounded monthly.

(a) What is the effective rate of interest?

(b) James wishes to accumulate $20,000 at the end of 8 years. How much must he deposit in the bank now?

3(a) Find the future value of an ordinary annuity with a payment of $3,000/monthly for 20 years at an interest rate of 3%/year compounded monthly.

(b) Find the present value of an ordinary annuity with a payment of $6,000/month for 15 years at an interest rate of 4%/year compounded monthly.

4(a) Tristan's pizza service requires a new van for its delivery service. In order to purchase the new van, the company borrows $60,000 from a bank. The loan is to be repaid in 7 years at the rate of 15% per year compounded monthly. How much is the regular monthly repayment for the van?

(b) Tristan decides to set up a sinking fund to purchase the van instead. The van is expected to cost $70,000 in 5 years' time. If the fund earns 10% interest per year compounded quarterly, determine the quarterly instalment Tristan should pay into the fund.

5. A group of private investors purchased a piece of land for $20 million. An initial down payment of 20% was made and they obtained financing for the balance. The term of the loan was 30 years and the interest rate was 4% per year compounded monthly. Find the monthly payment required.

6. Joanne has an initial sum of $5,000 in a bank account. She deposits $400 per month into the same account for the next 5 years. If the bank pays interest at the rate of 3% per year compounded monthly, how much will Joanne have in the account at the end of the 5 years? (Assume no withdrawals are made)

7. At the beginning of every month from January to October inclusive, Lucy makes a deposit of $100 into a savings fund that earns interest at the rate of 6% per year compounded monthly. If the fund pays interest at the end of each month, determine how much Lucy will have in the fund at the beginning of December.

BUSINESS MATHEMATICS

SESSION 7

DERIVATIVES AND ITS APPLICATION I

At the end of the session, students should be able to:

1. Compute limits of simple functions.
2. Understand the concept of continuous functions and their properties.
3. Compute the slope of a tangent line using first principles.
4. Use the basic rules of differentiation to compute the derivatives of polynomials.

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1. Introduction

Historically, the development of calculus in mathematics was initiated by Isaac Newton (1642 – 1727) and Gottfried Wilhelm Leibniz (1646 – 1716). It resulted from the investigation of two issues of which one was to find the tangent line to a curve at any given point of a curve. At first glance, this might seem irrelevant to any practical applications but mathematicians soon realized that finding the slope of a tangent line is mathematically equivalent to finding the rate of change of one quantity with respect to another. For example, problems related to the rate of change could include finding the distance covered by an object with respect to time or finding the rate of change of a company's profit with respect to the advertising expenditure.

The study of the slope of the tangent line led to the creation of a new field in mathematics, i.e. differential calculus, and the concept of the derivative of a function. Its development over the past centuries has made it an indispensable tool nowadays. Although we do not explicitly 'see' the mathematics, derivatives occur in almost all branches of pure and applied science, such as physics, engineering, economics, statistics and finance, to name a few. In the following chapters, we will focus on defining and computing derivatives followed by examining two important applications of derivatives: marginal functions and optimization.

2. Limits of simple functions

The derivative of a function is defined by a fundamental concept, that is the limit of a function.

2.1 Limit of a function

Consider the function defined by $f(x) = x + 2$. Suppose we are required to find the value of $f(x)$ when x approaches a fixed number 3. We can take a sequence of values of x approaching 3 from the left-hand side, e.g. 2.5, 2.9, 2.99, 2.999 or we can take a sequence of values of x approaching 3 from the right-hand side, e.g. 3.5, 3.1, 3.01, 3.001. We will obtain the results shown in Table 7.1.

Table 7.1: The limit of $f(x) = x + 2$ when x approaches 3

x	$f(x)$
2.5	4.5
2.9	4.9
2.99	4.99
2.999	4.999
2.9999	4.9999
2.99999	4.99999

x	$f(x)$
3.5	5.5
3.1	5.1
3.01	5.01
3.001	5.001
3.0001	5.0001
3.00001	5.00001

Note that in both cases, the value of $f(x)$ approaches 5 as x approaches 3. More importantly, the value of $f(x)$ will never reach 5; it just gets closer and closer to 5. Informally, we say that the limit of $f(x) = x + 2$ is 5 when x approaches 3 and we write it as

$$\lim_{x\to 3}(x+2)=5$$

The notion that $f(x)$ never reaches a particular value can be illustrated by another example. Suppose $g(x)=\dfrac{x^2-4}{x-2}$. Table 7.2 examines what happens to $g(x)$ when x approaches 2.

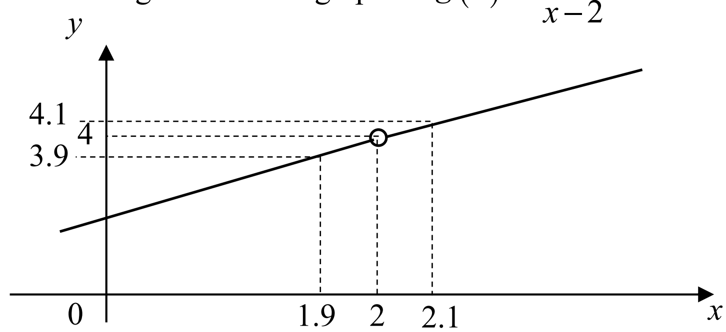
Table 7.2: The limit of $g(x)=\dfrac{x^2-4}{x-2}$ when x approaches 2

x	$g(x)$
1.5	3.5
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999

x	$g(x)$
2.5	4.5
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

The value of $g(x)$ approaches 4 when x approaches 2. However, x cannot take the value of 2 as $g(2)=\dfrac{2^2-4}{2-2}=\dfrac{0}{0}$ which is indeterminate. This is reflected in the graph of g (Figure 7.1).

Figure 7.1: The graph of $g(x)=\dfrac{x^2-4}{x-2}$



2.2 Evaluating the limit of a function

Consider a general case where the limit of a function *f* as *x* approaches the number *a* is *L*, i.e.:

$$\lim_{x \to a} f(x) = L$$

We are interested in computing the value of *L*. To evaluate the limits of functions algebraically, we make use of the notion that we can take *x* to be sufficiently close to the number *a* as well as the following properties of limits.

Table 7.3: Properties of limits

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

No.	Property
1.	$\lim_{x \to a} \left[f(x) \right]^r = \left[\lim_{x \to a} f(x) \right]^r = L^r$
2.	$\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) = cL$
3.	$\lim_{x \to a} f(x) \pm g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$
4.	$\lim_{x \to a} f(x) g(x) = \left[\lim_{x \to a} f(x) \right] \left[\lim_{x \to a} g(x) \right] = LM$
5.	$\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M}$ (provided $M \neq 0$)

For property 5, we might obtain the case where the limit of the function *g* is equal to 0. In this event, we say that the limit of the rational expression has an indeterminate form. However, we can still evaluate limits of this type using the following strategy.

1. Factorize and simplify the functions in the numerator and denominator of the rational functions.
2. Replace the original function with the simplified function. The simplified function should take on the same values as the original function everywhere except at $x = a$.
3. Evaluate the limit of the new function as *x* approaches *a*.

The following example illustrates the use of the properties of limits as well as this strategy to compute the limits of functions.

Example 7.1 Evaluate the following limits.

(a) $\lim_{x\to 5} 7x^2$ (b) $\lim_{x\to -2}\left(4x-x^4\right)$ (c) $\lim_{x\to 4}\left(5x\sqrt{x}+1\right)$

(d) $\lim_{x\to 1}\frac{x+5}{x+2}$ (e) $\lim_{x\to 7}\frac{x^2-49}{x-7}$ (f) $\lim_{x\to -4}\frac{2x^2-32}{x+4}$

Solution

(a) $\lim_{x\to 5} 7x^2 = 7\lim_{x\to 5} x^2 = 7\left(\lim_{x\to 5} x\right)^2 = 7(5)^2 = 175$

(b) $\lim_{x\to -2}\left(4x-x^4\right) = \lim_{x\to -2} 4x - \lim_{x\to -2} x^4 = 4(-2)-(-2)^4 = -24$

(c) $\lim_{x\to 4}\left(5x\sqrt{x}+1\right) = \lim_{x\to 4} 5x\sqrt{x} + \lim_{x\to 4} 1 = 5\left(\lim_{x\to 4} x\right)\left(\lim_{x\to 4}\sqrt{x}\right)+1 = 5(4)\left(\sqrt{4}\right)+1 = 41$

(d) $\lim_{x\to 1}\frac{x+5}{x+2} = \frac{\lim_{x\to 1}(x+5)}{\lim_{x\to 1}(x+2)} = \frac{6}{3} = 2$

(e) As $\lim_{x\to 7}(x-7)$ is 0, an indeterminate form is obtained. We will try to replace the original function through factorisation and simplification first.

Therefore, $\lim_{x\to 7}\frac{x^2-49}{x-7} = \lim_{x\to 7}\frac{(x+7)(x-7)}{x-7} = \lim_{x\to 7} x+7 = 14$

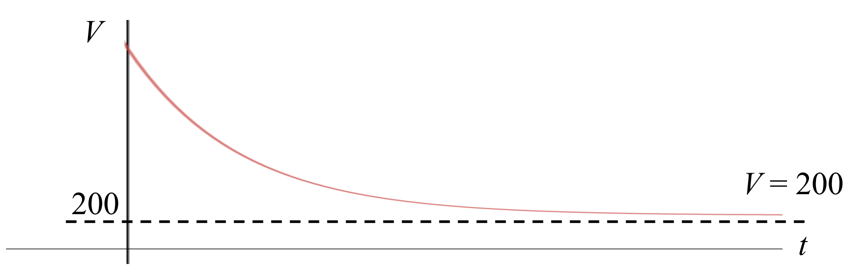
(f) As $\lim_{x\to -4}(x+4)$ is 0, an indeterminate form is obtained. We will try to replace the original function through factorisation and simplification first.

Therefore, $\lim_{x\to -4}\frac{2x^2-32}{x+4} = \lim_{x\to -4}\frac{2(x+4)(x-4)}{x+4} = \lim_{x\to -4} 2(x-4) = 2(-4-4) = -16$

2.3 Limits at infinity

What happens to a function *f* when *x* increases without bound? Certain situations might warrant the computation of the limit when *x* gets larger and larger. For example, we might want to know the scrap value of a piece of machine as time increases. An example is shown in Figure 7.2.

Figure 7.2: The value of an object when time increases



From the graph in Figure 7.2, we see that as time t increases without bound, the value V approaches 200. The line $V = 200$ is called a horizontal asymptote. In this instance, we say that the limit of a function f, when x approaches infinity, is 200.

When a function f has the limit L when x approaches infinity, i.e. increases without bound, we write

$$\lim_{x\to\infty} f(x) = L .$$

Similarly, when a function g has the limit M when x approaches negative infinity, i.e. decreases without bound, we write

$$\lim_{x\to-\infty} g(x) = M .$$

How we can compute the limit of a function at infinity algebraically? To do so, we will need the following results: When $n \geq 1$,

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Two examples are illustrated in Table 7.4. In both cases, the function approaches 0 when x approaches infinity.

Table 7.4: The limit of $f(x) = \frac{1}{x^n}$ when x approaches infinity for $n = 1$ and $n = 2$

x	$f(x) = \frac{1}{x}$
10	0.1
100	0.01
1000	0.001
10000	0.0001
100000	0.00001

x	$f(x) = \frac{1}{x^2}$
10	0.01
100	0.0001
1000	0.000001
10000	0.00000001
100000	1×10^{-10}

The following technique is often used to evaluate the limit at infinity of a rational expression:

1. Determine the highest power of x in the denominator of the expression, say x^n.
2. Divide all the terms in both the numerator and denominator by x^n.
3. Use the results above to evaluate the limit for each term.

The example below will illustrate this technique.

Example 7.2 Evaluate $\lim_{x\to\infty} \frac{x^2 + x + 1}{2x^2 - 8}$.

Solution

$$\lim_{x\to\infty}\frac{x^2+x+1}{2x^2-8}$$ [Note the highest power of the denominator, it is x^2 here]

$$=\lim_{x\to\infty}\frac{\frac{x^2}{x^2}+\frac{x}{x^2}+\frac{1}{x^2}}{\frac{2x^2}{x^2}-\frac{8}{x^2}}$$ [Divide every single term by the highest power,, i.e. x^2 here]

$$=\lim_{x\to\infty}\frac{1+\frac{1}{x}+\frac{1}{x^2}}{2-\frac{8}{x^2}}$$ [Simplify the expression]

$$=\frac{1+0+0}{2-0}$$ [Remember when x gets very large, both $\frac{1}{x}$ and $\frac{1}{x^2}$ approach 0]

$$=\frac{1}{2}$$

Example 7.3 Sierra's furniture store estimates that the average cost of making x chairs per year is given by the function $\overline{C}(x)=40+\frac{50000}{x}$. Evaluate $\lim_{x\to\infty}\overline{C}(x)$ and interpret your results.

Solution

$$\begin{aligned}\lim_{x\to\infty}\overline{C}(x)&=\lim_{x\to\infty}\left(40+\frac{50000}{x}\right)\\&=40+0\\&=40\end{aligned}$$

As the level of production x increases, the average cost approaches a constant value of \$40 per chair.

2.4 One-sided limits

Observe that in evaluating $\lim_{x\to a} f(x)$, x can approach the value a in two ways.

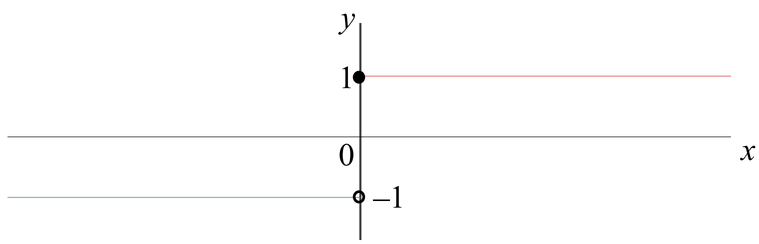
We can take x to be sufficiently close but always lesser than a. In this instance, we say x approaches a from the left hand side (of a number line) and use a^- to represent a number slightly smaller than a. The limit of the function as x approaches a^- is known as a left-hand limit and is denoted by $\lim_{x\to a^-} f(x)$.

Alternatively, we can take x to be sufficiently close but always greater than a. In this instance, we say x approaches a from the right hand side (of a number line) and use a^+ to represent a number slightly greater than a. The limit of the function as x approaches a^+ is known as a right-hand limit and denoted by $\lim_{x\to a^+} f(x)$.

We say that the limit of a function exists and is equal to L, i.e. $\lim_{x \to a} f(x) = L$, only when both the left-hand limit and the right-hand limit are both equal to L, i.e. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

Example 7.4 Let $f(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$. Does $\lim_{x \to 0} f(x)$ exist?

Solution
The graph of f is shown below.

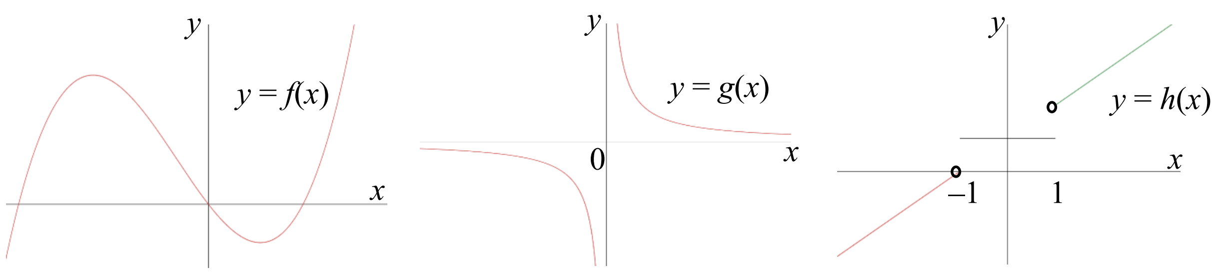


From the graph, we can see that $\lim_{x \to 0^-} f(x) = -1$ and $\lim_{x \to 0^+} f(x) = 1$. Since the one-sided limits are not equal, $\lim_{x \to 0} f(x)$ does not exist.

3. Continuous functions

Informally, a function is continuous at a point if the function at that point is devoid of holes, gaps or breaks. The graphs depicted in Figure 7.3 will illustrate the difference between continuous and discontinuous functions.

Figure 7.3: Continuous and discontinuous functions



f is continuous at all values of x.

g is discontinuous at the point $x = 0$. It is continuous everywhere else.

h is discontinuous at the points $x = -1$ and $x = 1$. It is continuous everywhere else.

Formally, a function f is continuous at a number $x = a$ if the following conditions are satisfied:

1. $f(a)$ is defined,
2. $\lim_{x \to a} f(x)$ exists and
3. $\lim_{x \to a} f(x) = f(a)$.

In other words, a function f is continuous at the point $x = a$ if the limit of f at $x = a$ exists and is equal to $f(a)$. From this definition, we can show that the constant function $f(x) = c$ and the identity function $f(x) = x$ are continuous at all values of x.

Furthermore, we can infer certain properties of continuous functions using the properties of limits. Suppose that f and g are continuous at the point $x = a$, then the following functions are also continuous at that point: (a) $[f(x)]^n$ (where n is a real number), (b) $f \pm g$, (c) fg and f/g (provided that $g(a) \neq 0$).

From these properties, we can identify some functions that are continuous:

1. A polynomial function $y = f(x)$ is continuous at all values of x and
2. A rational function $y = f(x)/g(x)$ is continuous at all values of x except where $g(x) = 0$.

Example 7.5 Find the values of x for which each function is continuous.

(a) $f(x) = 3x^4 - x + 8$ (b) $g(x) = \dfrac{5x}{x^2+1}$ (c) $h(x) = \dfrac{7x}{x^2-1}$

Solution

(a) Since $f(x) = 3x^4 - x + 8$ is a polynomial function, it is continuous for all values of x.

(b) The function $g(x)$ is a rational function. However, note that the denominator $x^2 + 1$ can never be 0. Hence $g(x)$ is continuous for all values of x.

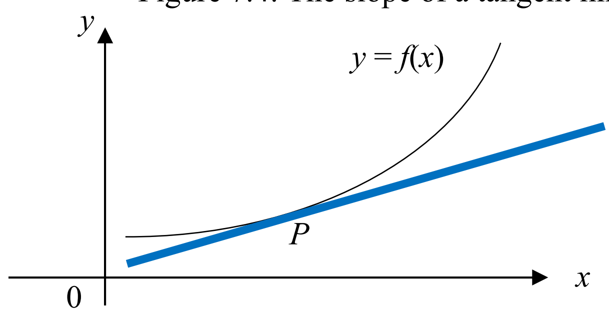
(c) The function $h(x)$ is a rational function. Observe that $x^2 - 1 = (x + 1)(x - 1)$ and therefore the denominator is equal to 0 when $x = 1$ or -1. This implies that $h(x)$ is continuous for all values of x except where $x = 1$ or -1.

4. Slope of a tangent line

4.1 Computing the slope of a tangent line using first principles

Recall that the initial premise of calculus was to find the rate of change of a quantity with respect to another and this is akin to finding the slope of a curve. Graphically, the slope of a curve at a given point is equivalent to the slope of the tangent line, as shown in Figure 7.4.

Figure 7.4: The slope of a tangent line

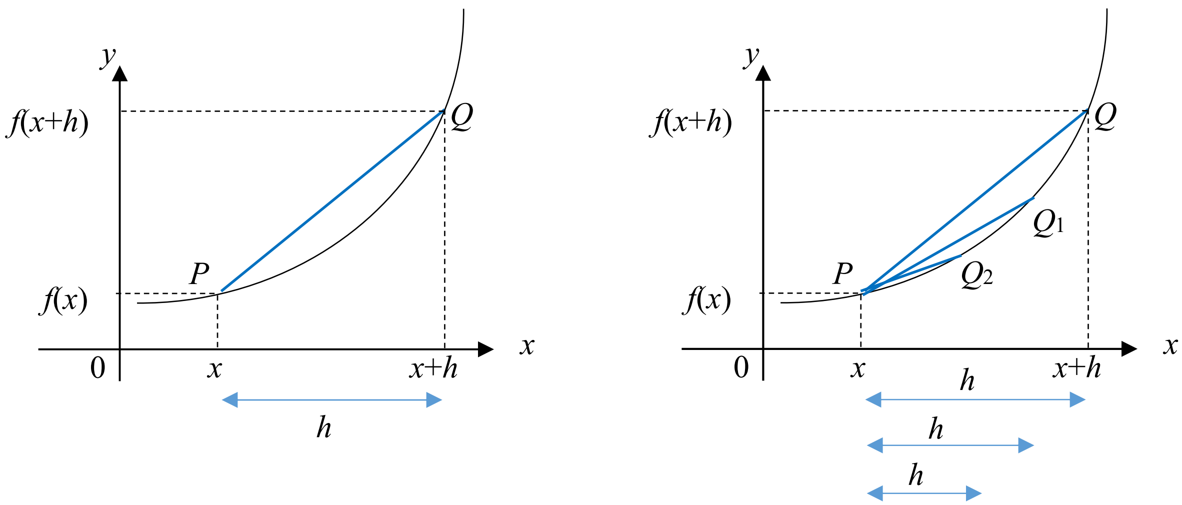


The tangent line to the curve cuts the curve at only one point.

Slope of tangent line at P = slope of curve at that point P

In order to find the slope of the tangent line, we will use the formula to compute the slope of a straight line together with the properties of limits.

Figure 7.5: Computing the slope of a tangent line



(a) Computing the slope of a secant line PQ

(b) Approximating PQ to the tangent line by letting h approach zero

In Figure 7.5(a), we want to find the slope of the tangent at point P with coordinates $(x, f(x))$. Choose a point Q on the curve that has its x-coordinate a small distance h from P, i.e. $x + h$. The corresponding y-coordinate will thus be $f(x+h)$. The straight line connecting P to Q is known as a secant line. As PQ is a straight line, we can compute the slope of PQ using the formula for slope in Chapter 3.

$$\text{Slope of } PQ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

In Figure 7.5(b), observe that we can make Q approach P along the curve by decreasing the value of h. Observe that the value of h gets closer and closer to zero. This will cause the secant line PQ to get closer and closer to the tangent line to P. In other words, as h approaches zero, the line PQ approaches the tangent line to P. When h gets very close to zero, we can say that the slope of PQ approximates the slope of the tangent line to P.

Therefore, the limit of the slope of PQ when h approaches zero is the slope of the tangent line to the graph of $y = f(x)$ at the point $P(x, f(x))$. This limit is known as the derivative of f at x and is denoted by $f'(x)$. This leads us to the following formula.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 7.6 Find the slope of the tangent line to the graph of $f(x) = x^2$. What is the slope of the curve at the point where $x = 3$?

Solution
The slope of the tangent line at any point on the graph is given by the derivative $f'(x)$.
Since $f(x) = x^2$, we have $f(x + h) = (x + h)^2$. Thus

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} \\
&= \lim_{h\to 0}\frac{x^2+2hx+h^2-x^2}{h} \\
&= \lim_{h\to 0}\frac{2hx+h^2}{h} \\
&= \lim_{h\to 0}\frac{h(2x+h)}{h} \\
&= \lim_{h\to 0}(2x+h) \\
&= 2x
\end{aligned}$$

At the point where $x = 3$, $f'(x) = 2(3) = 6$.

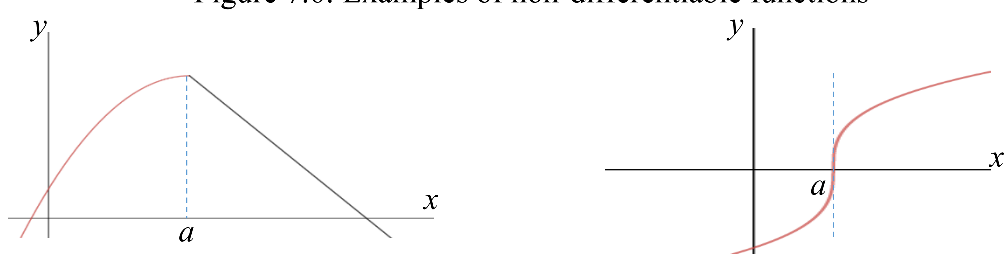
As the slope of the tangent line at any point on the curve is equivalent to the slope of the curve at that same point, we can say that the slope of the curve at $x = 3$ is 6.

4.2 Differentiability

It is possible that the limit examined in the above section might not exist. When the limit does exist, i.e. the derivative of a function exists, we say that the function is differentiable.

Conversely, a function is not differentiable for certain values of x if the derivative of the function does not exist for those values. These functions usually have an abrupt change of direction at a point or have a tangent line where the slope is vertical. Some examples of these are illustrated in Figure 7.6. In each case, the function is non-differentiable at $x = a$.

Figure 7.6: Examples of non-differentiable functions



5. Basic rules of differentiation

Although the method used in Section 2.4 for deriving the slope of the curve is based on first principles, it is tedious even for simple functions. Since then, mathematicians have derived certain rules that will simplify this process.

Before we examine these rules, note that this process of finding the derivative of functions is called differentiation and we will use $f'(x)$ to mean the derivative of a function f with respect to x. Thus, to indicate that we are differentiating f with respect to x, we will write

$$f'(x) = \frac{d}{dx}\left[f(x)\right]$$

Rule 1: If c is a constant, then $\frac{d}{dx}(c) = 0$.

Recall that the graph of $f(x) = c$ is a straight line parallel to the x-axis. Geometrically, the tangent line to a straight line coincides with the line itself. Hence, the slope of $f(x) = c$ must be zero.

Rule 2 (The Power Rule): If n is a real number, then $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

We have verified Rule 2 in the case where $n = 2$ in Example 7.6 and the general case will not be proved here. Note that if the function involves a radical or a rational expression, we first rewrite the function using fractional or negative powers before using the Power Rule.

Rule 3: If c is a real number and f is a differentiable function, then $\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right]$.

The derivative of a constant times a differentiable function is equal to the constant times by the derivative of the function. Note that c must be a constant and not another function of x.

Rule 4: If f and g are functions, then $\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$.

The derivative of the sum (or difference) of two differentiable functions is equal to the sum (or difference) of their derivatives. This result can be extended to the sum (or difference) of any finite number of differentiable functions.

These basic rules of differentiation are summarised in Table 7.5.

Table 7.5: Basic rules of differentiation

Rule	Formula	Example
1.	$\frac{d}{dx}(c)=0$	$\frac{d}{dx}(5)=0$
2.	$\frac{d}{dx}(x^n)=nx^{n-1}$	$\frac{d}{dx}(x^7)=7x^{7-1}=7x^6$
3.	$\frac{d}{dx}[cf(x)]=c\frac{d}{dx}[f(x)]$	$\frac{d}{dx}(2x^7)=2\frac{d}{dx}(x^7)=2(7x^6)=14x^6$
4.	$\frac{d}{dx}[f(x)\pm g(x)]=\frac{d}{dx}[f(x)]\pm\frac{d}{dx}[g(x)]$	$\frac{d}{dx}(x^7+5)=\frac{d}{dx}(x^7)+\frac{d}{dx}(5)=7x^6$

Example 7.7 Find the derivatives of the following functions.

(a) $f(x)=\sqrt{x}+\frac{3}{x}$ (b) $g(x)=x(x^3+1)$ (c) $h(x)=\frac{5x^2+6x-1}{x}$

Solution

(a) We first rewrite the terms in terms of fractional or negative powers,

$$f(x)=x^{\frac{1}{2}}+3x^{-1}$$

Applying the basic rules, we have

$$\begin{aligned}
f'(x)&=\frac{d}{dx}\left(x^{\frac{1}{2}}+3x^{-1}\right)\\
&=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)+\frac{d}{dx}\left(3x^{-1}\right) && \text{[Rule 4]}\\
&=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)+3\frac{d}{dx}\left(x^{-1}\right) && \text{[Rule 3]}\\
&=\frac{1}{2}x^{\frac{1}{2}-1}+3(-1)x^{-1-1} && \text{[Rule 2]}\\
&=\frac{1}{2}x^{-\frac{1}{2}}-3x^{-2} \quad or \quad \frac{1}{2\sqrt{x}}-\frac{3}{x^2}
\end{aligned}$$

(b) If possible, we simplify the function before differentiating,

$$\begin{aligned}
g(x)&=x(x^3+1)\\
&=x^4+x
\end{aligned}$$

Applying the basic rules, we have

$$
\begin{aligned}
g'(x) &= \frac{d}{dx}\left(x^4 + x\right) \\
&= \frac{d}{dx}\left(x^4\right) + \frac{d}{dx}\left(x^1\right) && \text{[Rule 4]} \\
&= 4x^{4-1} + (1)x^{1-1} && \text{[Rule 2]} \\
&= 4x^3 + x^0 && [x^0 = 1] \\
&= 4x^3 + 1
\end{aligned}
$$

(c) If possible, we simplify the function before differentiating,

i.e. $h(x) = \dfrac{5x^2 + 6x - 1}{x} = \dfrac{5x^2}{x} + \dfrac{6x}{x} - \dfrac{1}{x} = 5x + 6 - \dfrac{1}{x} = 5x + 6 - x^{-1}$

Applying the basic rules, we have

$$
\begin{aligned}
h'(x) &= \frac{d}{dx}\left(5x + 6 - x^{-1}\right) \\
&= 5\frac{d}{dx}\left(x^1\right) + \frac{d}{dx}(6) - \frac{d}{dx}\left(x^{-1}\right) && \text{[Rule 4 and 3]} \\
&= 5(1)x^{1-1} + 0 - (-1)x^{-1-1} && \text{[Rule 2 and 1]} \\
&= 5x^0 + x^{-2} && [x^0 = 1] \\
&= 5 + x^{-2} \quad or \quad 5 + \frac{1}{x^2}
\end{aligned}
$$

Example 7.8 Find the slope and an equation of the tangent line to the graph of $f(x) = x^3 + 7$ at the point (1, 8).

Solution

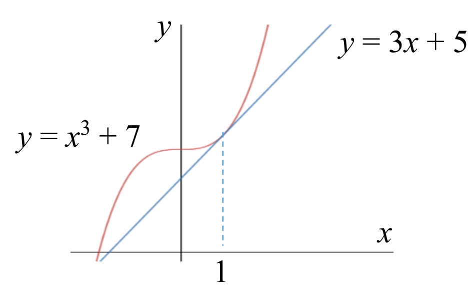
The slope of the tangent line at any point on the graph is given by the derivative $f'(x)$.

$$f'(x) = \frac{d}{dx}\left(x^3 + 7\right) = \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}(7) = 3x^2 + 0 = 3x^2$$

The slope of the tangent line at the point (1, 8) (where $x = 1$) is $f'(1) = 3(1)^2 = 3$.

Since the tangent line is a straight line, we can use the point slope form to obtain its equation.

$$
\begin{aligned}
y - 8 &= 3(x - 1) && [y - y_1 = m(x - x_1)] \\
y - 8 &= 3x - 3 \\
y &= 3x + 5
\end{aligned}
$$



6. Discussion questions

1. Find the limit of the following functions.

(a) $\lim_{x\to 1}\left(2x^2+3\right)$ (b) $\lim_{x\to 0}\frac{e^x+e^{2x}}{4}$

(c) $\lim_{x\to 3}\frac{x-3}{x^2-9}$ (d) $\lim_{x\to 7}\frac{x^2-49}{x-7}$

(e) $\lim_{x\to \infty}\frac{x^2+1}{x^2-2}$ (f) $\lim_{x\to \infty}\frac{4x^3+8x^2-1}{2x^3-2x}$

2. Let $f(x)=\begin{cases}3, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$. Determine if $\lim_{x\to 0} f(x)$ exist.

3. Find the values of x for which each function is continuous.

(a) $7x^3-x^2+23x+1$ (b) $\frac{x^2-2x+1}{x^2+7x+10}$

4. Compute the derivative of the function $f(x) = 2x + 7$ from first principles. Is the answer expected?

5. Find the derivative of the following functions.

(a) $f(x)=x^5$ (b) $f(x)=\frac{3}{x}$

(c) $f(x)=4\sqrt{x}-x$ (d) $f(x)=\frac{7}{x^3}+\frac{1}{\sqrt{x}}$

(e) $f(t)=(t-3)^2$ (e) $f(w)=\frac{2w^3-4w+3}{w}$

6. The function f is defined by $f(x) = 10x^3$.
(a) Find the derivative of the function, $f'(x)$.
(b) Find the slope of the tangent line at the point (1, 10).
(c) Find the equation of the tangent line at the point (1, 10).

7. Supplementary questions

1. Find the limit of the following functions.

(a) $\lim_{x \to 2} \frac{3^x - 7}{5}$ (b) $\lim_{x \to 2} \left(2^x + 1\right)^x$

(c) $\lim_{x \to -5} \frac{x+5}{x^2 - 25}$ (d) $\lim_{x \to -2} \frac{x^2 + 4x + 4}{x+2}$

(e) $\lim_{x \to \infty} \frac{3x^2 + x + 11}{5x^2 - 2x + 1}$ (f) $\lim_{x \to \infty} \frac{x^3 + 3}{x^2 - 1}$

2. Let $f(x) = \begin{cases} x & x < 1 \\ 0 \quad if & x = 1 \\ -x+2 & x > 1 \end{cases}$. Determine if $\lim_{x \to 1} f(x)$ exist. If it exists, what is $\lim_{x \to 1} f(x)$?

3. Find the values of x for which each function is continuous.

(a) $-x^4 + x - 11$ (b) $\frac{x^2 + x - 6}{x^2 + 1}$

4. Compute the derivative of the function $f(x) = x^2 - 2x + 1$ from first principles. Hence find the value of x where the slope of the tangent to $f(x)$ is 0.

5. Find the derivative of the following functions.

(a) $f(x) = 0.01x^2 - 0.5x + 0.02$ (b) $f(x) = x\sqrt{x}$

(c) $f(x) = (x+2)(x-2)$ (d) $f(x) = \frac{x^3 - 2x^2 + 1}{x^2}$

6. The function f is defined by $f(x) = x^3 - 3x + 1$.

(a) Find the derivative of the function, $f'(x)$.

(b) Find the value(s) of x where the slope of f is 0.

BUSINESS MATHEMATICS

SESSION 8

DERIVATIVES AND ITS APPLICATION II

At the end of the session, students should be able to:

1. Use the product rule, quotient rule, chain rule and general power rule to compute the derivatives of complex functions.
2. Compute derivatives of exponential and logarithmic functions.
3. Understand and derive marginal cost, revenue and profit functions.
4. Find the relative extrema of a function.

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1. Introduction

This chapter follows on from the previous chapter in exploring more advanced techniques of differentiation. The basic rules posited in Chapter 7 can only be used to find the derivative of simple functions and composite functions will require the use of more complex rules. For that purpose, we will examine the product rule, quotient rule and chain rule here. In addition, we will also compute the derivatives of exponential and logarithmic functions, both of which are used extensively in economics and business. Finally, we will examine how to derive marginal functions in economics as well as find the relative extrema of functions.

2. Product rule, Quotient rule, Chain rule and General Power rule

2.1 Product rule and Quotient rule

Product rule: If *f* and *g* are functions, then

$$\frac{d}{dx}\left[f(x)g(x)\right]=f(x)\frac{d}{dx}\left[g(x)\right]+g(x)\frac{d}{dx}\left[f(x)\right].$$

The product rule is used to differentiate the product of two functions and can be extended to the product of any finite number of functions. Note that the derivative of $f(x)g(x)$ is NOT $f'(x)g'(x)$.

Quotient rule: If *f* and *g* are functions, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}\left[f(x)\right]-f(x)\frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}.$$

The quotient rule is used to differentiate rational expressions provided that the denominator is not zero. Similar to the product rule, note that the derivative of $f(x)/g(x)$ is NOT $f'(x) / g'(x)$.

Example 8.1 Find the derivative of the following functions.

(a) $f(x)=(x^2+6)(3x-1)$ (b) $g(x)=\dfrac{2x}{x-1}$

Solution

(a) As f is a product of two functions, we will use the product rule.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[(x^2+6)(3x-1)\right] \\
&= (x^2+6)\frac{d}{dx}(3x-1)+(3x-1)\frac{d}{dx}(x^2+6) \qquad \text{[Product rule]} \\
&= (x^2+6)(3)+(3x-1)(2x) \\
&= 3x^2+18+6x^2-2x \\
&= 9x^2-2x+18
\end{aligned}$$

(b) As g is a rational expression, we will use the quotient rule.

$$\begin{aligned}
g'(x) &= \frac{d}{dx}\left(\frac{2x}{x-1}\right) \\
&= \frac{(x-1)\frac{d}{dx}(2x)-(2x)\frac{d}{dx}(x-1)}{(x-1)^2} \qquad \text{[Quotient rule]} \\
&= \frac{(x-1)(2)-(2x)(1)}{(x-1)^2} \\
&= \frac{2x-2-2x}{(x-1)^2} \\
&= \frac{-2}{(x-1)^2}
\end{aligned}$$

In general, as the product and quotient rules are quite complicated, we use them when the function is not easily simplified. For example, it is not necessary to use the product and quotient rule for the functions below:

$$\frac{d}{dx}\left[(x+1)(x-1)\right]=\frac{d}{dx}(x^2-1)=2x$$

$$\frac{d}{dx}\left(\frac{x^2-x}{x}\right)=\frac{d}{dx}\left(\frac{x^2}{x}-\frac{x}{x}\right)=\frac{d}{dx}(x-1)=1$$

2.2 Chain rule and General Power rule

Chain rule: If h is the composite of two functions f and g, then

$$h'(x) = \frac{d}{dx} g\left[f(x)\right] = g'\left[f(x)\right] f'(x).$$

The chain rule is used extensively to differentiate the composition of two functions. However, very often, the chain rule is used to formulate a secondary formula that will help us differentiate specific types of function directly, e.g. $[f(x)]^n$,

General Power rule: If n is a real number and f is a function, then

$$\frac{d}{dx}\left[f(x)\right]^n = n\left[f(x)\right]^{n-1} f'(x).$$

The general power rule is a corollary of the chain rule where $g(x) = x^n$ and $g'(x) = nx^{n-1}$. It helps us differentiate powers of functions without the hassle of expanding the functions.

Example 8.2 Find the derivative of $f(x) = (2x^2 + 1)^{10}$.

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x^2+1\right)^{10} \\ &= 10\left(2x^2+1\right)^{10-1}\left[\frac{d}{dx}\left(2x^2+1\right)\right] \qquad \text{[General Power rule]} \\ &= 10\left(2x^2+1\right)^9(4x) \\ &= 40x\left(2x^2+1\right)^9 \end{aligned}$$

3. Derivatives of exponential and logarithm functions

3.1 Exponential functions

The derivative of an exponential function with base e (≈ 2.7182818) is equal to itself, i.e.

$$\frac{d}{dx}\left(e^x\right) = e^x.$$

Note that this result is only true when the base is e and the exponent is x. It is not true for a general exponential function with base b, i.e. b^x.

What if the exponent is another function f? To differentiate such functions, we employ the Chain rule to obtain another formula.

$$\frac{d}{dx}\left[e^{f(x)}\right]=e^{f(x)}f'(x)$$

As a side note, if we wish to differentiate an exponential function with a base other than e, we would have to convert it to the form $e^{f(x)}$, e.g. $2^x = e^{(\ln 2)x}$. However, this is outside the scope of this book.

Example 8.3 Find the derivative of the following functions.

(a) $f(x)=3e^x+x^3$ (b) $g(x)=e^{3x+1}$

Solution

(a) As f is a sum of two functions, we will differentiate each term separately (sum rule).

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[3e^x+x^3\right] \\ &= 3\frac{d}{dx}\left(e^x\right)+\frac{d}{dx}\left(x^3\right) \\ &= 3e^x+3x^2 \end{aligned}$$

(b) Since the exponent is a function, we will use the chain rule form for exponential functions.

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(e^{3x+1}\right) \\ &= e^{3x+1}\left[\frac{d}{dx}(3x+1)\right] \\ &= e^{3x+1}(3) \\ &= 3e^{3x+1} \end{aligned}$$

3.2 Logarithm functions

The derivative of a logarithm function with base e is equal to the reciprocal of x, i.e.

$$\frac{d}{dx}\left(\ln|x|\right)=\frac{1}{x}.$$

Similar to exponential functions, this result is only true when the base is e and the function "inside" is x. It is not true for a general logarithm function with base b, i.e. $\log_b x$.

What if the "inside" function is another function f? To differentiate such functions, we once again employ the Chain rule to obtain another formula.

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{f'(x)}{f(x)} \quad or \quad \frac{1}{f(x)}\left[f'(x)\right]$$

If we wish to differentiate a logarithm function with a base other than e, we would have to change its base to e. However, this is outside the scope of this book.

Example 8.4 Find the derivative of the following functions.

(a) $f(x) = \ln\left(x^2+1\right)$ (b) $g(x) = \ln\left(\sqrt{x}e^{-x^3}\right)$

Solution

(a) Since there is a function "inside", we will use the chain rule form for logarithm functions.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\ln\left(x^2+1\right)\right] \\ &= \frac{1}{x^2+1}\left[\frac{d}{dx}\left(x^2+1\right)\right] \\ &= \frac{1}{x^2+1}(2x) \\ &= \frac{2x}{x^2+1} \end{aligned}$$

(b) At times, it is necessary to use the laws of logarithms to first simplify the given expression.

$$\begin{aligned} g(x) &= \ln\left(\sqrt{x}e^{-x^3}\right) \\ &= \ln\sqrt{x} + \ln e^{-x^3} && [\log AB = \log A + \log B] \\ &= \ln x^{\frac{1}{2}} + \ln e^{-x^3} \\ &= \frac{1}{2}\ln x - x^3 \ln e && [\log A^n = n\log A] \\ &= \frac{1}{2}\ln x - x^3 && [\ln e = \log_e e = 1] \end{aligned}$$

Hence,

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(\frac{1}{2}\ln x - x^3\right) \\ &= \frac{1}{2}\frac{d}{dx}\ln x - \frac{d}{dx}x^3 \\ &= \frac{1}{2x} - 3x^2 \end{aligned}$$

3.3 Miscellaneous examples of derivatives

Sometimes, the function requires us to use a combination of rules to compute the derivative. For example, we might need to use the chain rule together with the product rule to differentiate the function.

Example 8.5 Find the derivative of the following functions.

(a) $f(x) = \left(\dfrac{x}{x+2}\right)^4$ (b) $g(x) = e^{4x}\ln x$

Solution

(a) We will need to use the general power rule here. However, as the function "inside" is a rational expression, we would also need the quotient rule.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{x}{x+2}\right)^4 \\
&= 4\left(\frac{x}{x+2}\right)^{4-1}\left[\frac{d}{dx}\left(\frac{x}{x+2}\right)\right] && \text{[General Power rule]} \\
&= 4\left(\frac{x}{x+2}\right)^{3}\left[\frac{(x+2)(1)-x(1)}{(x+2)^2}\right] && \text{[Quotient rule]} \\
&= 4\left(\frac{x}{x+2}\right)^{3}\left[\frac{2}{(x+2)^2}\right] \\
&= \frac{8x^3}{(x+2)^5}
\end{aligned}$$

(b) As the function is a product of two expressions, we will need to use the product rule together with the rules for exponential and logarithm functions.

$$\begin{aligned}
g'(x) &= \frac{d}{dx}\left(e^{4x}\ln x\right) \\
&= e^{4x}\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}\left(e^{4x}\right) && \text{[Product rule]} \\
&= \left(e^{4x}\right)\left(\frac{1}{x}\right) + (\ln x)\left(e^{4x}\right)\left(\frac{d}{dx}4x\right) \\
&= \frac{e^{4x}}{x} + (\ln x)\left(e^{4x}\right)(4) \\
&= \frac{e^{4x}}{x} + 4e^{4x}\ln x
\end{aligned}$$

4. Marginal functions in Economics

Marginal analysis is the study of the rate of change of economic quantities. For example, a person might be interested in both the absolute value of a country's gross domestic product as well as the rate at which it is growing or declining. A businessman could also be interested in the how much the cost of a product changes with respect to the level of production. We will begin by examining marginal cost functions.

4.1 Cost functions

In Chapter 4, we have encountered the total cost function, $C(x)$, which gives the total cost required to manufacture x units of the product. The marginal cost function is defined to be the derivative of the total cost function and gives the cost incurred in producing an additional unit of a product.

$$\text{Marginal cost function} = C'(x)$$

The average cost of producing x units of a product is computed by dividing the total cost by the number of units produced. It is denoted by $\overline{C}(x)$ (read "C bar of x").

$$\text{Average cost function} = \overline{C}(x) = \frac{C(x)}{x}$$

Accordingly, the marginal average cost function is defined to be the derivative of the average cost function and measures the rate of change of the average cost function with respect to the number of units produced.

$$\text{Marginal average cost function} = \overline{C}'(x)$$

Example 8.6 Debbie's Electric Company manufactures tablet PCs. The company determined that the weekly total cost of producing x tablet PCs is given by the function

$$C(x) = 6000 + 200x - 0.1x^2$$

(a) What is the actual cost incurred for manufacturing the 201^{st} tablet PC?
(b) Find the marginal cost function when $x = 200$ and interpret your results.
(c) Find the average cost function when $x = 200$ and interpret your results.
(d) Find the marginal average cost function and interpret your results.

Solution

(a) We will use the total cost function to compute the actual cost here.

$$\begin{aligned}\text{Total cost to manufacture 201 units} &= C(201)\\ &= 6000 + 200(201) - 0.1(201)^2\\ &= \$42{,}159.90\end{aligned}$$

$$\begin{aligned}\text{Total cost to manufacture 200 units} &= C(200)\\ &= 6000+200(200)-0.1(200)^2\\ &= \$42,000.00\end{aligned}$$

$$\begin{aligned}\text{Cost to manufacture 201}^{\text{st}}\text{ unit} &= C(201)-C(200)\\ &= 42,159.90-42,000.00\\ &= \$159.90\end{aligned}$$

(b) The marginal cost function is obtained by differentiating the cost function.

$$\begin{aligned}C'(x) &= \frac{d}{dx}\left(6000+200x-0.1x^2\right)\\ &= 200-0.2x\end{aligned}$$

$$\begin{aligned}C'(200) &= 200-0.2(200)\\ &= \$160\end{aligned}$$

Marginal cost is the cost needed to produce an additional unit, i.e. \$160 is approximately the cost needed to produce the 201st unit.

(c) The average cost function is obtained by dividing the total cost function with the quantity.

$$\begin{aligned}\overline{C}(x) &= \frac{C(x)}{x}\\ &= \frac{6000+200x-0.1x^2}{x}\\ &= \frac{6000}{x}+200-0.1x\\ \overline{C}(200) &= \frac{6000}{200}+200-0.1(200)\\ &= \$210\end{aligned}$$

Each unit of tablet PC costs an average of \$210 to make. It is higher than the variable cost as it takes into account fixed costs.

(d) The marginal average cost function is obtained by differentiating the average cost function.

$$\begin{aligned}\overline{C}'(x) &= \frac{d}{dx}\left(\frac{6000}{x}+200-0.1x\right)\\ &= -\frac{6000}{x^2}-0.1\end{aligned}$$

Note that the marginal average cost function here is always negative. This means that as the level of production x increases, the average cost per unit decreases.

4.2 Revenue and Profit functions

Recall that the revenue function gives the total revenue realised from the sale of x units and it is given by $R(x) = px$ where p is the selling price per unit. However, the selling price of a product is dependent on its demand in the market that it operates in. The relationship between the unit selling price (p) and the quantity (x) demanded is through the demand equation (see Chapter 4). Thus, if the demand equation is given by $p = f(x)$,

$$\text{Revenue function, } R(x) = px = xf(x)$$

The marginal revenue function is defined as the derivative of the revenue function and gives the revenue realized from an additional unit of a product.

$$\text{Marginal revenue function} = R'(x)$$

The profit function is the difference between the total revenue and total cost. The marginal profit function is defined as the derivative of the profit function and gives the profit realized from an additional unit of a product.

$$\text{Profit function, } P(x) = R(x) - C(x)$$

$$\text{Marginal profit function} = P'(x)$$

Example 8.7 A company has determined that the total cost function in manufacturing one unit of television set is given by $C(x) = 100x + 200{,}000$. The monthly demand for the laptop is given by the equation $p = -0.02x + 400$.

(a) Find the revenue function and the marginal revenue function.

(b) Find the marginal profit function. Interpret the result when $x = 1000$.

Solution

(a) Revenue function, $R(x) = px = (-0.02x + 400)x = -0.02x^2 + 400x$

Marginal revenue function, $R'(x) = \dfrac{d}{dx}\left(-0.02x^2 + 400x\right) = -0.04x + 400$

(b) Profit function,
$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= -0.02x^2 + 400x - (100x + 200{,}000) \\ &= -0.02x^2 + 300x - 200{,}000 \end{aligned}$$

Marginal profit function, $P'(x) = \dfrac{d}{dx}\left(-0.02x^2 + 300x - 200{,}000\right) = -0.04x + 300$

When $x = 1000$, $P'(1000) = -0.04(1000) + 300 = 340$

The profit realized from the 1001st unit is \$340.

5. Relative extrema

This section is a first step in analysing the properties of functions to solve optimization problems. We will first look at the definition of increasing and decreasing functions, followed by determining the relative extrema of functions.

5.1 Increasing and decreasing functions

A function f is said to be increasing on an interval (a, b) if for any two numbers x_1 and x_2 in (a, b), $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. Informally, this means that as x increases, $f(x)$ increases, i.e. the rate of change of f with respect to x is positive.

Conversely, a function f is said to be decreasing on an interval (a, b) if for any two numbers x_1 and x_2 in (a, b), $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. Informally, this means that as x increases, $f(x)$ decreases, i.e. the rate of change of f with respect to x is positive.

Since the derivative measures the slope of the tangent line which is equivalent to the rate of change, we have the following results:

If $f'(x) > 0$ for every x in the interval (a, b), then f is increasing on (a, b).

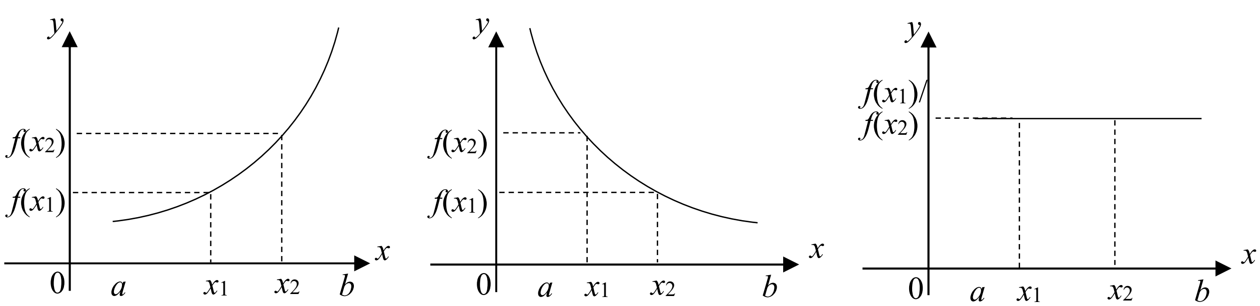
If $f'(x) < 0$ for every x in the interval (a, b), then f is decreasing on (a, b).

What happens to the function when $f'(x) = 0$? When the derivative of a function is zero, the rate of change is zero, i.e. f does not change when x changes. Thus,

If $f'(x) = 0$ for every x in the interval (a, b), then f is constant on (a, b).

Figure 8.1 shows the difference between these functions.

Figure 8.1: Increasing and decreasing functions



(a) f is increasing. $f'(x) > 0$ for every point in the interval (a, b).

(b) f is decreasing. $f'(x) < 0$ for every point in the interval (a, b).

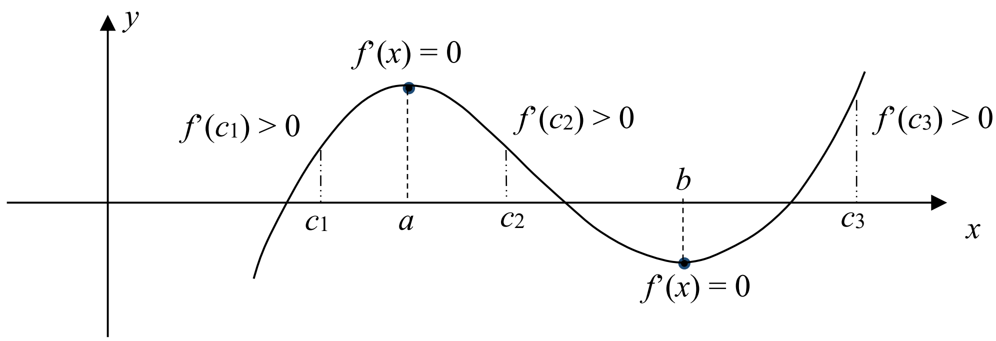
(c) f is constant. $f'(x) = 0$ for every point in the interval (a, b).

In most cases, a function can be increasing and decreasing at various intervals. If we wish to find these intervals where a function is increasing or decreasing, we can use the following technique:

1. Find all values of x for which $f'(x) = 0$ or $f'(x)$ does not exist.
2. The values of x in 1) will form open intervals.
3. Select a test value c in each interval in 2) and determine the sign of $f'(c)$ in that interval.
4. If (i) $f'(c) > 0$, then the function is increasing
   (ii) $f'(c) < 0$, then the function is decreasing
   (iii) $f'(c) = 0$, then the function is constant in the interval.

   Figure 8.2 will illustrate this technique.

Figure 8.2: Determining intervals of a function that are increasing or decreasing



Example 8.8 Determine the intervals where the function $f(x) = x^3 - 3x^2 - 24x + 32$ is increasing or decreasing.

Solution

$$f'(x) = \frac{d}{dx}\left(x^3 - 3x^2 - 24x + 32\right)$$
$$= 3x^2 - 6x - 24$$

Step 1:

$$f'(x) = 0$$
$$3x^2 - 6x - 24 = 0$$
$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-24)}}{2(3)}$$
$$x = -2 \quad or \quad 4$$

Step 2:
The x values found in step 1 divide the number line into the intervals $(-\infty, -2)$, $(-2, 4)$ and $(4, \infty)$.

Step 3:
Pick any number in the interval to substitute into $f'(x)$ and check for its sign. The results are shown in the table below.

Interval	Test point, c	$f'(c)$	Sign
$(-\infty,-2)$	-3	$f'(-3)=3(-3)^2-6(-3)-24=21$	$f'(c)>0$
$(-2,4)$	0	$f'(0)=3(0)^2-6(0)-24=-24$	$f'(c)<0$
$(4,\infty)$	5	$f'(5)=3(5)^2-6(5)-24=21$	$f'(c)>0$

From the table, f is increasing in both the intervals $(-\infty, -2)$ and $(4, \infty)$ but decreasing in the interval $(-2, 4)$.

5.2 Relative extrema

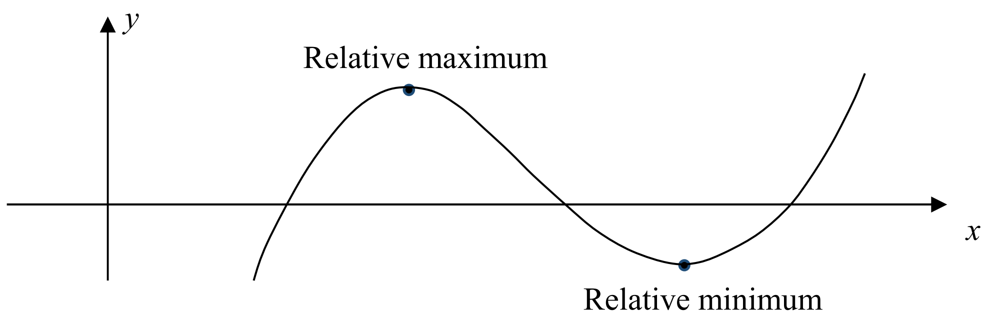
Suppose f is a function defined on an interval (a, b) and c is a number in the interval.

The function f has a relative maximum at c if $f(x) \leq f(c)$ for all x in (a, b).

The function f has a relative minimum at c if $f(x) \geq f(c)$ for all x in (a, b).

Informally, a relative maximum implies that the value of $f(x)$ is the highest point within the interval and a relative minimum implies that the value of $f(x)$ is the lowest point within the interval. Together, the relative maximum and minimum are known as the relative extrema of the function. Figure 8.3 gives an example of a function with its relative extrema.

Figure 8.3: The relative extrema of a function



Observe that the slope of the tangent line to the curve at the left of a relative maximum is positive and that the slope at the right is negative. To change from positive to negative, the slope must pass through a point in which its value is zero, i.e. the slope of the tangent line is parallel to the x-axis or $f'(x) = 0$.

A similar argument can be made for the relative minimum of a curve. Hence we can conclude that

At the relative extrema (maximum or minimum) of a function f, $f'(x) = 0$.

Now that we have a way to determine the relative extrema of a function, we will need a technique to find out whether it is a relative maximum or minimum. One procedure we can use is the First Derivative Test, which is as follows:

1. First determine the x-coordinate of the relative extrema by setting $f'(x) = 0$.
2. Check the sign of $f'(x)$ to the left and right of each relative extrema.
   (i) If $f'(x)$ changes from positive to negative, it is a relative maximum.
   (ii) If $f'(x)$ changes from negative to positive, it is a relative minimum.
   (iii) If there is no change in sign, it is not a relative extrema.
3. Compute the relative maximum or minimum by finding the value of $f(x)$ at that point.

Example 8.9 Find and determine the nature of the relative extrema of the function $f(x) = x^3 - 3x^2 - 24x + 32$.

Solution

$$f'(x) = 3x^2 - 6x - 24$$
$$f'(x) = 0$$
$$3x^2 - 6x - 24 = 0$$
$$x = -2 \;\; or \;\; 4 \quad \text{[from Example 8.8]}$$

To determine whether the points are relative maximum or minimum, we draw a table and choose a value of x to the left and right of the relative extrema. We then calculate the slope of the curve at those values, i.e. substitute the x values into $f'(x)$.

x	-3	-2	-1
$f'(x)$	21	0	-15
Slope	/	—	\

x	3	4	5
$f'(x)$	-15	0	21
Slope	\	—	/

At $x = -2$: A relative maximum occurs since $f'(x)$ goes from positive to negative.

Relative maximum $= f(-2)$
$= (-2)^3 - 3(-2)^2 - 24(-2) + 32$
$= 60$

At $x = 4$: A relative minimum occurs since $f'(x)$ goes from negative to positive.

Relative maximum $= f(4)$
$= (4)^3 - 3(4)^2 - 24(4) + 32$
$= -48$

6. Discussion questions

1. Find the derivative of the following functions.

(a) $f(x) = (x^3 - 1)(2x + 1)$ (b) $f(x) = \dfrac{x+3}{x-2}$

(c) $g(x) = (x^2 + 15)^{20}$ (d) $g(x) = x\sqrt{(2x+1)}$

2. Find the derivative of the following functions.

(a) $f(x) = e^{7x+1}$ (b) $g(x) = \ln 7x$

(c) $h(x) = e^{2x+3} - \ln(x+1)$ (d) $f(x) = e^x(2x+1)$

(e) $g(x) = x(\ln x)$ (f) $h(x) = \dfrac{5x-2}{e^x}$

3. The function f is given by $f(x) = e^{4x-1}$.

(a) Find the derivative of the function, $f'(x)$.

(b) Find the slope of the function at the point $x = 1$.

4. Execs Furniture Pte Ltd makes a line of executive desks for companies. The management has estimated that the total cost per year for making x units of their Junior Executive model is given by $C(x) = 50x + 500{,}000$ dollars. Find the marginal average cost function.

5. The monthly demand for a PC manufacturer is given by $p = 600 - 0.05x$, where p denotes the wholesale unit price in dollars and x denotes the quantity demanded. The monthly total cost function associated with manufacturing a PC is given by

$$C(x) = 0.000002x^3 - 0.03x^2 + 400x + 80000.$$

(a) Find the revenue function $R(x)$ and the profit function $P(x)$.

(b) Find the marginal cost function, marginal revenue function and marginal profit function.

(c) Compute the marginal profit function when $x = 2000$ and interpret your results.

6. Find the interval(s) where each function is increasing and where it is decreasing.

(a) $f(x) = 5x + 7$ (b) $g(x) = x^3 - x$

7. Find the relative maximum and/or relative minimum of the following functions.

(a) $f(x) = x^2 - 4x$ (b) $g(x) = x^3 - 3x^2 + 5$

7. Supplementary questions

1. Find the derivative of the following functions.

(a) $f(x)=(x^2-1)(7-x^3)$ (b) $f(x)=\dfrac{5-2x}{5x+2}$

(c) $g(x)=(1-2x+3x^2)^{12}$ (d) $g(x)=x\sqrt{(1-4x)}$

2. Find the derivative of the following functions.

(a) $f(x)=e^{5x+1}\ln x$ (b) $f(x)=\dfrac{3x^2-1}{e^x}$

(c) $f(x)=(e^{2x}+\ln x^3)^4$ (d) $f(x)=\left[\ln(3x^2-1)\right]^5$

3. The number of fitness trackers demanded each month is related to the unit price by the equation $p=\dfrac{50}{0.01x^2+1}$, where p is measured in dollars and x is in units of a thousand. Find the revenue function and marginal revenue function.

4. A division of Debbie's Electric Company manufactures cameras. The quantity x of cameras demanded each month is given by $p = -0.006x + 190$, where p is the wholesale unit price in dollars. The monthly total cost associated with manufacturing a camera is given by

$$C(x)=0.000002x^3-0.02x^2+120x+70000.$$

(a) Find the marginal cost function, marginal revenue function and marginal profit function.
(b) Compute the marginal profit function when $x = 2000$ and interpret your results.

5. Find the interval(s) where each function is increasing and where it is decreasing.

(a) $f(x)=x^2-1$ (b) $g(x)=x^3+9x^2+11$

6. Find the relative maximum and/or relative minimum of the following functions.

(a) $f(x)=3x^2-6x+17$ (b) $g(x)=4x^2+\dfrac{5832}{x}+5$

BUSINESS MATHEMATICS

SESSION 9

DERIVATIVES AND ITS APPLICATION III

At the end of the session, students should be able to:

1. Find higher order derivatives of a function.
2. Apply the second derivative to determine the intervals of concavity of a function and hence establish the nature of a relative extrema.
3. Solve optimization problems using derivatives in business and economics.

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1. Introduction

This chapter follows on from the previous chapter in examining techniques to solve optimization problems. Optimization problems usually involves computing the maximum or minimum of a quantity and this is used frequently in science, engineering, finance, economics etc. For example, an electrical engineer might want to find out the maximum current that can occur in a circuit or a businessman might want to compute the number of units that he should manufacture to maximise his profits. One basic method of solving these problems requires the use of first and second order derivatives and we will investigate this method in greater detail here.

2. Higher order derivatives

2.1 First and second order derivatives

When we differentiate a function f, the derivative f' is known as the first order derivative of f. As f' is also a function, we might also consider the derivative of f'. The derivative of f' is known as the second order derivative of f (as it is computed by differentiating f twice) and is denoted by f''.

What does f'' measure? Since f' measures the rate of change of f with respect to x, by a parallel association, f'' measures the rate of change of f' with respect to x, i.e. it measures how fast the slope of a function is changing.

Example 9.1 Find the first and second order derivative of $f(x) = x^3 - 2x^2 + 6x - 13$.

Solution

First order derivative: $f'(x) = \dfrac{d}{dx}\left[f(x)\right] = 3x^2 - 4x + 6$

Second order derivative: $f''(x) = \dfrac{d}{dx}\left[f'(x)\right] = 6x - 4$

2.2 Higher order derivatives

We can extend the above argument to higher order derivatives, that is, the third, fourth, fifth, etc. order derivatives of f. The notation is respectively f^3, f^4, f^5, etc.

Example 9.2 Find the third order derivative of $f(x) = e^{2x}$.

Solution

First order derivative: $$f'(x) = \frac{d}{dx}\left[f(x)\right] = 2e^{2x}$$

Second order derivative: $$f''(x) = \frac{d}{dx}\left[f'(x)\right] = 2\left(2e^{2x}\right) = 4e^{2x}$$

Third order derivative: $$f^3(x) = \frac{d}{dx}\left[f''(x)\right] = 4\left(2e^{2x}\right) = 8e^{2x}$$

3. Applications of the second derivative

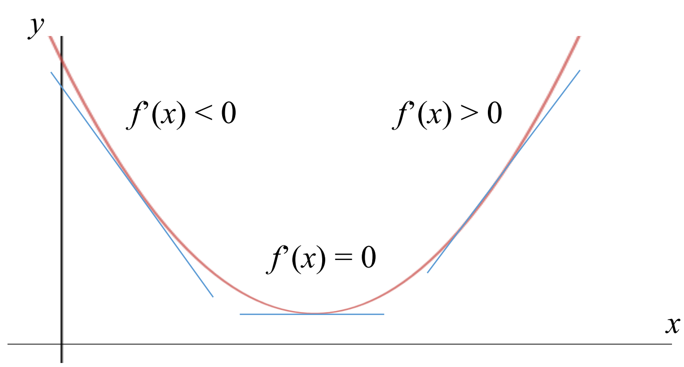
In this section, we will investigate two applications of the second derivative: 1) To determine the intervals of concavity of a function and 2) to establish the nature of relative extrema, i.e. whether it is a relative maximum or a minimum.

3.1 Concavity of a function

The graph of a function f is defined to be concave upwards on an interval (a, b) if f' is increasing on (a, b). An example is shown in Figure 9.1. Notice that as x increases, the slope of the tangent line (that is, the slope of the curve) goes from a negative value to a positive value. This implies that the derivative f' increases as x increases. If we recall that the second order derivative f'' measures the rate of change of f', this suggests that:

If the graph is concave upwards, f'' is positive, i.e. $f''(x) > 0$.

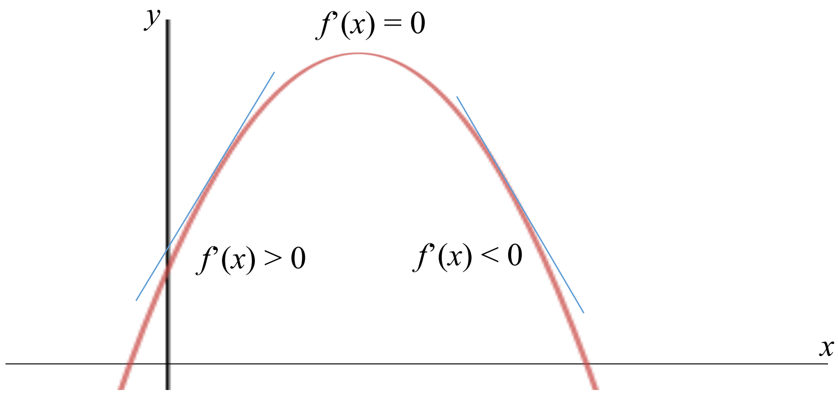
Figure 9.1: A curve that is concave upwards



Conversely, the graph of a function f is defined to be concave downwards on an interval (a, b) if f' is decreasing on (a, b). An example is shown in Figure 9.2. From the curve, we can observe that as x increases, the slope of the curve goes from a positive value to a negative value. This implies that the derivative f' decreases as x increases. Again, recall that the second order derivative f'' measures the rate of change of f'. This suggests that:

If the graph is concave downwards, f'' is negative, i.e. $f''(x) < 0$.

Figure 9.2: A curve that is concave downwards

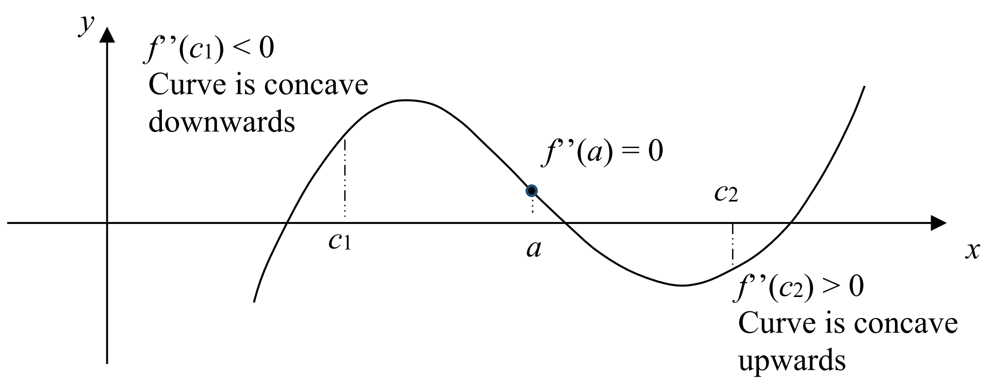


As the sign of f'' indicates whether the curve is concave upwards or downwards, this provides us with a procedure to determine the concavity of a curve at different intervals, which is as follows:

1. Find all values of x for which $f''(x) = 0$ or $f''(x)$ does not exist.
2. The values of x in 1) will form open intervals.
3. Select a test value c in each interval in 2) and determine the sign of $f''(c)$ in that interval.
4. If (i) $f''(c) > 0$, then the curve is concave upwards
   (ii) $f''(c) < 0$, then the curve is concave downwards in the interval
   (iii) $f''(c) = 0$, then the curve is neither concave upwards or downwards.

This procedure is very similar to the procedure described in the previous chapter to find intervals of a function that are increasing or decreasing. Figure 9.3 illustrates this procedure.

Figure 9.3: Determining intervals of concavity of functions



Example 9.3 Determine the intervals where the graph of the function $f(x) = x^3 - 3x^2 - 24x + 32$ is concave upwards and where it is concave downwards.

Solution

$$f'(x) = 3x^2 - 6x - 24$$
$$f''(x) = 6x - 6$$

Step 1:

$$f''(x) = 0$$
$$6x - 6 = 0$$
$$x = 1$$

Step 2:
The x-values found in step 1 divide the number line into the intervals $(-\infty, 1)$ and $(1, \infty)$.

Step 3:
Pick a number in the intervals to substitute into $f''(x)$ and check for its sign. The results are shown in the table below.

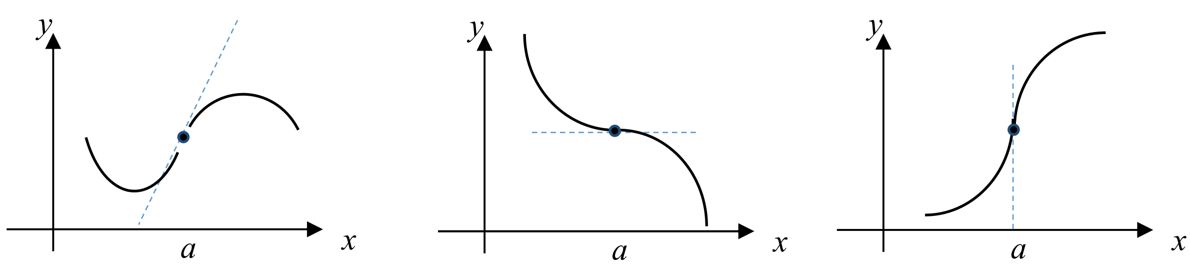
Interval	Test point, c	$f''(c)$	Sign
$(-\infty, 1)$	0	$f''(0) = 6(0) - 6 = -6$	$f''(c) < 0$
$(1, \infty)$	2	$f''(2) = 6(2) - 6 = 6$	$f'(c) > 0$

From the table, f is concave downwards in the interval $(-\infty, 1)$ but concave upwards in the interval $(1, \infty)$.

3.2 Inflection points

The point on the graph where the concavity changes is known as a point of inflection or inflection point. Figure 9.4 shows typical examples of inflection points.

Figure 9.4: Examples of inflection points



To find points of inflection of a function, we adopt a similar procedure described in the previous chapter on determining the relative extrema of a function.

1. Determine the x-coordinate of the inflection point by setting $f''(x) = 0$ or where $f''(x)$ does not exist.
2. Check the sign of $f''(x)$ to the left and right of each number found in 1.
   (i) If $f''(x)$ changes from positive to negative, the concavity changes and the point found is an inflection point.
   (ii) If $f''(x)$ changes from negative to positive, the concavity changes and the point found is an inflection point.
   (iii) If there is no change in sign, there is no change in concavity. The point found is not an inflection point.

Example 9.4 Find the inflection point(s) of the function $f(x) = x^3 - 3x^2 - 24x + 32$.

Solution
As in the previous example, $f'(x) = 3x^2 - 6x - 24$ and $f''(x) = 6x - 6$.

$$f''(x) = 0$$
$$6x - 6 = 0$$
$$x = 1$$

To determine whether the point is an inflection point, we draw a table and choose a value of x to the left and right of the point. We then calculate the value of $f''(x)$ at those values.

x	0	1	2
$f''(x)$	–6	0	6
Shape	Concave downwards		Concave upwards

An inflection point occurs at $x = 1$ as the concavity of the curve changes.
Since $f(1) = (1)^3 - 3(1)^2 - 24(1) + 32 = 6$, inflection point = (1, 6)

Example 9.5 The total sales S of bathtubs manufactured by Babbly Corporation is related to the amount of money x the company spends on advertising its products by the formula

$$S(x) = -0.01x^3 + 1.5x^2 + 200, \qquad (0 \le x \le 100)$$

where x and S are in thousands of dollars. Find the inflection point and intervals of concavity of S. Interpret your results.

Solution

$$S'(x) = -0.03x^2 + 3x$$
$$S''(x) = -0.06x + 3$$
$$S''(x) = 0$$
$$-0.06x + 3 = 0$$
$$x = 50$$

A possible inflection point occurs at $x = 50$.

We draw a table to determine whether it is an inflection point. The table will also help us determine the concavity of the function at the different intervals.

x	49	50	51
$S''(x)$	0.06	0	–0.06
Shape	Concave upwards		Concave downwards

An inflection point occurs at $x = 50$ as the concavity of the curve changes.

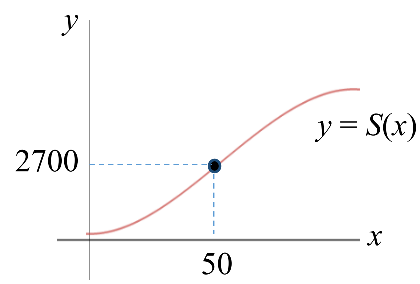
$$S(50) = -0.01(50)^3 + 1.5(50)^2 + 200$$
$$= 2700$$

Inflection point = (50, 2700).

Since $0 \leq x \leq 100$, we can conclude that S is concave upwards in the interval (0, 50) but concave downwards in the interval (50, 100). The graph of $y = S(x)$ is shown in the figure to the right.

In the interval (0, 50), S is concave upwards: The total sales increases slowly at first but as more money is spent, the total sales increases rapidly.

In the interval (50, 100), S is concave downwards: Any additional money spent on advertising results in increased sales but at a slower rate.



Thus, in this example, the inflection point (50, 2700) represents the point of diminishing returns.

3.3 Relative maximum and minimum

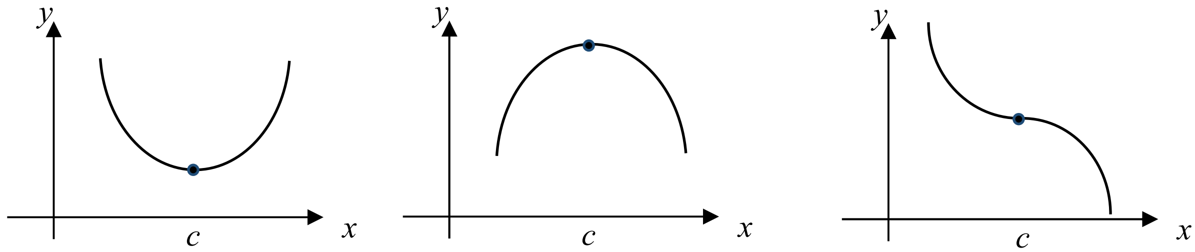
In chapter 8, we established whether a relative extremum is a maximum or minimum by applying first derivative test. The test can get quite tedious, especially if there were many relative extrema in the function. By computing the second derivative, we can use a second method to determine the nature of the relative extrema, i.e. the Second Derivative Test.

Observe that at a relative minimum of f, the graph is necessarily concave downwards. On the other hand, at a relative maximum of f, the graph is necessarily concave upwards. Thus, by finding the concavity of the function at the relative extrema, we are able to determine its nature with the following rules (See Figure 9.5).

Suppose f is a function and a relative extremum occurs at $x = c$.

> If $f''(c) > 0$, then the point is a relative minimum.
> If $f''(c) < 0$, then the point is a relative maximum.
> If $f''(c) = 0$ or $f''(c)$ does not exist, then it is inconclusive. We will need the First Derivative Test to determine the nature of the relative extremum.

Figure 9.5: The Second Derivative Test



(a) $f''(c) > 0$. $f(c)$ is a relative minimum.

(b) $f''(c) > 0$. $f(c)$ is a relative maximum.

(c) $f''(c) = 0$. We do not know if $f(c)$ is a relative minimum or maximum.

Example 9.6 Find the relative maximum and minimum of the function $f(x) = x^3 - 12x$.

Solution

$$f'(x) = 3x^2 - 12$$
$$f''(x) = 6x.$$

To find the coordinates of the relative extremum, we let $f'(x) = 0$.

$$3x^2 - 12 = 0$$
$$x^2 = 4$$
$$x = 2 \ \ or \ \ -2$$

To show whether the relative extrema are maximum or minimum, we find the sign of $f''(c)$.

At $x = 2$: $f''(2) = 6(2) = 12 > 0$. Hence a relative minimum occurs at $x = 2$.
Relative minimum = $f(2) = 2^3 - 12(2) = -16$

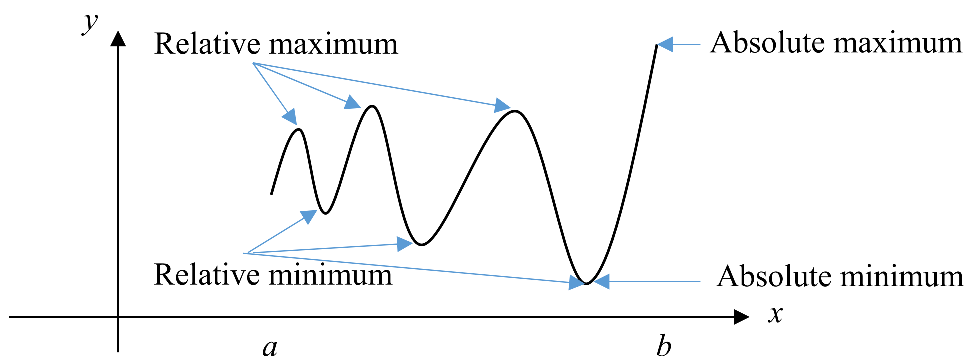
At $x = -2$: $f''(-2) = 6(-2) = -12 < 0$. Hence a relative maximum occurs at $x = -2$.
Relative maximum = $f(-2) = (-2)^3 - 12(-2) = 16$

4. Optimization problems

4.1 Absolute extrema

Consider a function f with a domain of $[a, b]$ and a graph as shown in Figure 9.6. In the graph, f has several relative maximum and relative minimum. However, there is a single point on the graph that has the greatest value of $f(x)$. Likewise, there is also a single point on the graph that has the least value of $f(x)$. These points are known as the absolute maximum and absolute minimum respectively.

Figure 9.6: Absolute extrema of a function



More precisely,

The function f has an absolute maximum at c if $f(x) \leq f(c)$ for all x in the domain of f.

The function f has an absolute minimum at c if $f(x) \geq f(c)$ for all x in the domain of f.

Notice that in Figure 9.6, the absolute minimum occurs at a relative extrema whereas the absolute maximum occurs at the end point of a closed interval. It can be shown that if the domain of a continuous function is a closed interval, then it is guaranteed to have both an absolute maximum and absolute minimum value. Furthermore, any absolute extrema will occur either at the relative extrema or at the endpoints of the domain. This leads us to the following procedure for finding the absolute extrema of a function in a closed interval.

Suppose f is a continuous function defined on the closed interval $[a, b]$.

1. First determine the x-coordinate of the relative extrema by setting $f'(x) = 0$.
2. Compute the value of $f(x)$ at each relative extrema and the endpoints of the interval, i.e. $f(a)$ and $f(b)$.
3. The absolute maximum and absolute minimum will correspond to the greatest and least value in Step 2.

What if the domain is an open interval? In that instance, it could either have zero, one or two absolute extremum. Moreover, any absolute extremum must occur at a relative extremum and we may have to apply the second derivative test to show that it is an absolute maximum or minimum.

Example 9.7 Find the absolute maximum and minimum of the function $f(x) = x^2$, (a) when f is defined on the interval $[-2, 3]$ and (b) when f is defined for $x > -2$.

Solution

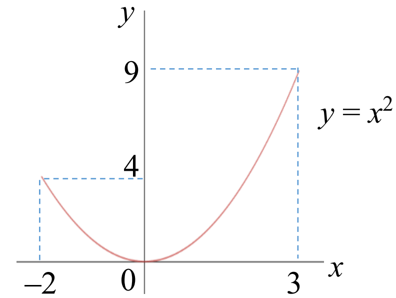
$$f'(x) = 2x$$

To find relative extremum, we set $f'(x) = 0$.

$$2x = 0$$

$$x = 0$$

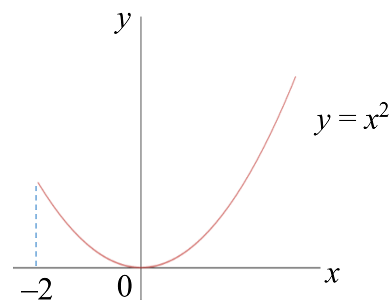
(a) Since f is defined on a closed interval, we check values of $f(x)$ at the relative extremum and endpoints.



$$\text{At } x=0, f(0)=0^2=0$$
$$\text{At } x=-2, f(-2)=(-2)^2=4$$
$$\text{At } x=3, f(3)=3^2=9$$

Comparing the different values of $f(x)$, it follows that :
Absolute maximum is 9 when $x = 3$ and
Absolute minimum is 0 when $x = 0$.

(b) Since f is defined on an open interval and $x = 0$ is in the interval, we use the second derivative test to check the nature of the extremum and the shape of the function.



$$f''(x)=2>0$$

Hence $x = 0$ gives a relative minimum.

Since $f''(x) > 0$ for all values of x in the domain, the graph is concave upwards everywhere in the domain. Thus the absolute minimum is $f(0)= 0$ when $x = 0$.
There is no absolute maximum.

4.2 Optimization problems in business and economics

The previous mathematical principles on relative and absolute extrema can be applied to problems pertaining to business and economics, e.g. finding the maximum profit or minimum average costs.

Example 9.8 Debbie's Electric Company estimates that the total profit (in dollars) realized from manufacturing and selling x units of their holographic television set is given by

$$P(x)=-0.02x^2+300x-100,000, \qquad (0\le x\le 10,000)$$

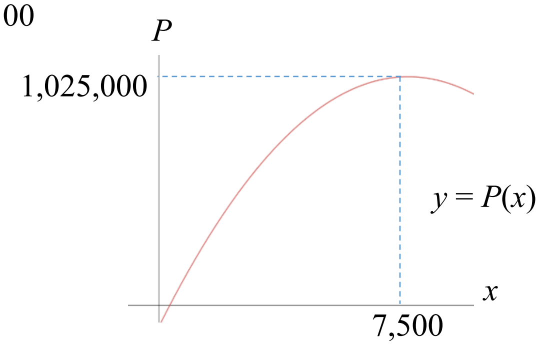
How many units of the holographic television set must the company sell to maximize its profits?

Solution
Since P is defined on a closed interval, we find the absolute maximum of P on [0, 10,000].

$$P'(x)=-0.04x+300$$

To find the relative extremum, we set $P'(x) = 0$.

$$-0.04x+300=0$$
$$0.04x=300$$
$$x=7500$$



We check values of P at the relative extremum and the endpoints.

$$\text{At } x = 7,500, P(7500) = 1,025,000$$

$$\text{At } x = 0, P(0) = -100,000$$

$$\text{At } x = 10,000, P(10000) = 900,000$$

From comparing the above computations, we see that the company will realize a maximum profit of \$1,025,000 by producing 7,500 units.

Example 9.9 The daily total costs incurred by Dolly's Drones Pte Ltd for producing x number of Model Z drones is given by the function

$$C(x) = 0.0025x^2 + 80x + 10,000, \qquad x > 0$$

How many units of the drone must the company produce to minimize its average cost? What is the minimum average cost?

Solution

$$\text{Average cost}, \overline{C}(x) = \frac{0.0025x^2 + 80x + 10,000}{x}$$

$$= 0.0025x + 80 + \frac{10000}{x}$$

$$\overline{C}'(x) = 0.0025 - \frac{10000}{x^2}$$

Setting $\overline{C}'(x) = 0$,

$$0.0025 - \frac{10000}{x^2} = 0$$

$$x^2 = \frac{10000}{0.0025}$$

$$= 4000000$$

$$x = 2000 \quad or \quad -2000$$

Since x cannot be negative, we reject $x = -2000$. To show that this is a relative minimum, we use the second derivative test.

$$\overline{C}''(x) = \frac{20000}{x^3}$$

$$\overline{C}''(2000) = \frac{20000}{(2000)^3} > 0$$

Hence, a relative minimum occurs at $x = 2000$. Furthermore, since $\overline{C}''(x) > 0$ for all $x > 0$, the graph is concave upwards everywhere, so the relative minimum is also the absolute minimum.

$$\overline{C}(2000) = 0.0025(2000) + 80 + \frac{10000}{2000} = 90$$

The minimum average cost is \$90 per unit.

Example 9.10 The present value of the retail price of an office unit in the central business district is given by

$$P(t) = 500,000e^{-0.08t+\frac{1}{3}\sqrt{t}}, \qquad (0 \le t \le 10)$$

Find the optimal present value of the unit's retail price.

Solution

The optimal present value of *P* is given by the absolute maximum of *P* on [0, 10]

$$P'(t) = 500,000e^{-0.08t+\frac{1}{3}\sqrt{t}} \frac{d}{dt}\left(-0.08t + \frac{1}{3}\sqrt{t}\right)$$

$$= 500,000e^{-0.08t+\frac{1}{3}\sqrt{t}}\left(-0.08 + \frac{1}{6\sqrt{t}}\right)$$

Setting $P'(t) = 0$, we have

$$500,000e^{-0.08t+\frac{1}{3}\sqrt{t}}\left(-0.08 + \frac{1}{6\sqrt{t}}\right) = 0$$

Since $500,000e^{-0.08t+\frac{1}{3}\sqrt{t}} > 0$ for all values of *t*,

$$-0.08 + \frac{1}{6\sqrt{t}} = 0$$

$$\frac{1}{\sqrt{t}} = 0.48$$

$$\sqrt{t} = \frac{1}{0.48}$$

$$t = \left(\frac{1}{0.48}\right)^2$$

$$= 4.34$$

We check values of *P* at the relative extremum and the endpoints.

At $t = 4.34, P(4.34) = 500,000e^{-0.08(4.34)+\frac{1}{3}\sqrt{4.34}} = 707,565.58$

At $t = 0, P(0) = 500,000e^{-0.08(0)+\frac{1}{3}\sqrt{0}} = 500,000$

At $t = 10, P(10) = 500,000e^{-0.08(10)+\frac{1}{3}\sqrt{10}} = 644,645.56$

From comparing the above computations, we see that the optimal present value is $707,565.58 and this will occur 4.34 years from now.

5. Discussion questions

1. Find the second and third order derivatives of the following functions.

(a) $f(x) = x^7 - 6x^4 + 2$ (b) $g(x) = e^{3x-1}$

2. Find the inflection point and the intervals of concavity for the following functions.

(a) $f(x) = 6x^3 - 18x^2 + 12x - 20$ (b) $g(x) = x^4 - 2x^3 + 6$

3. Find the relative maximum and/or minimum of the following functions.

(a) $f(x) = 2x^2 + 3x + 7$ (b) $g(x) = x + \dfrac{9}{x}$

4. Find the absolute maximum and/or minimum of the following functions.

(a) $f(x) = x^3 + 3x^2 - 1$ on $[-1, 5]$ (b) $g(x) = x - \ln x$ on $\left[\dfrac{1}{2}, 3\right]$

5. The management of Sparkly's Juice has estimated that the profits (in dollars) realized from the weekly production and sale of x bottles of sparkling apple juice is given by

$$P(x) = -0.000002x^3 + 6x - 400 .$$

What is the largest profit that the company can make in a week?

6. The total cost function for producing a certain type of dress is $C(x) = 0.5(0.09x^2 + 190)$, where x represents the number of dresses produced. Find the level of production that will minimize the average cost. Round the answer to the nearest integer.

7. A manufacturer of baseball bats finds that the total cost of producing x baseball bats per day is given by the function

$$C(x) = 400 + 4x + 0.0001x^2 .$$

Furthermore, the manufacturer finds that the selling price of each racket p is related to x by the demand equation $p = 10 - 0.0004x$. If all the baseball bats that are manufactured can be sold, find the maximum profit per day for the manufacturer.

6. Supplementary questions

1. Find the second and third order derivatives of the following functions.

(a) $f(x) = 2x^{10} + 3x^3 - 2x$ (b) $g(x) = \ln(5x+2)$

2. The sales (in billions of dollars) in restaurants and bars in California from 1993 ($t = 0$) to 2000 ($t = 7$) can be approximated by the function

$$S(t) = 0.195t^2 + 0.32t + 23.7, \qquad (0 \le t \le 7)$$

(a) Show that the function is increasing in the interval [0, 7]. Interpret your results.
(b) Show that the function is concave upwards in the interval [0,7]. Interpret your results.

3. Find the relative maximum and/or minimum of the following functions.

(a) $f(x) = x^2 + \dfrac{2}{x}$ (b) $g(x) = e^{x^2-2x}$

4. The office space rents at the MIS building through the years 2010 (t = 0) to 2015 (t = 5) is approximated by the function

$$R(t) = -0.701t^3 + 3.61t^2 + 0.7t + 36.5, \qquad (0 \le t \le 5)$$

where $R(t)$ is the price per square foot in dollars. What was the highest office space rent at the MIS building during this period?

5. A division of a Korean firm manufactures a particular type of tablet PC in which the total cost for producing x tablet PCs per month is

$$C(x) = 0.000008x^3 - 0.07x^2 + 30x + 20{,}000.$$

The selling price of each tablet PC is related to x by the demand equation

$$p = -0.08x + 180$$

Find the level of production that will yield a maximum profit for the manufacturer.

6. The total cost function for producing x bottles of sweet sauce is given by the function

$$C(x) = 0.0000002x^3 + 5x + 400$$

(a) Find the level of production that results in the smallest average cost.
(b) Find the level of production for which the average cost is equal to the marginal cost.
(c) Compare your results in (a) and (b).

7. The weekly quantity demanded of a certain type of uniform is related to the unit price p by the demand equation

$$p = \sqrt{700 - x}$$

where p is in dollars and x is the number of uniforms made. To maximize the revenue, how many uniforms should be manufactured on a weekly basis? Give your answer to the nearest integer.

BUSINESS MATHEMATICS

SESSION 10

INTEGRATION AND ITS APPLICATION I

At the end of the session, students should be able to:

1. Find indefinite integrals of polynomials and exponential functions.
2. Solve simple initial value problems.
3. Compute integrals by using substitution.

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1. Introduction

Whilst differential calculus focuses on investigating the rate of change of quantities with respect to another, integral calculus centres on the exact opposite problem. If the rate of change is known, is it possible to find the relation between the two quantities? For example, if we know the rate of consumption of electricity with respect to time, can we determine how much electricity is needed over the course of 5 years to meet the needs of a society? The principal technique that will be studied is the antiderivative of a function and the process of integration to find the antiderivative. In the following chapters, we will examine this process followed by some of its applications on business and economics.

2. Indefinite integrals of functions

2.1 Antiderivatives

Suppose we were to find the derivative of $f(x) = x^2$. Using the rules of differentiation in the previous chapters, we would obtain

$$\frac{d}{dx}\left(x^2\right) = 2x .$$

We define $2x$ as the derivative of x^2. In a similar fashion, we define x^2 as the antiderivative of $2x$. If we extend the example to general functions, we have the following definition:

A function F is an antiderivative of f on an interval if $F'(x) = f(x)$ for all x in the interval.

However, when we use the above definition strictly, we immediately run into a problem. Consider the function $f(x) = 2x$ in the earlier example. An antiderivative could be $F(x) = x^2 + 1$. It could also be $G(x) = x^2 + 2$ or $H(x) = x^2 - 5$ or $J(x) = x^2 - 8$ etc. This is because when we differentiate the functions F, G, H or J, we obtain the function $f(x) = 2x$. Following this line of reasoning, we see that there are infinitely many antiderivatives. Therefore, we conclude that if $F'(x) = f(x)$, then every antiderivative F on an interval must be of the form $F(x) + c$, where c is a constant.

2.2 The indefinite integral

The process of finding the antiderivative of a function is known as integration. We write

$$\int f(x)\,dx = F(x) + c$$

The symbol $\int$ is used to indicate the operation of integration and is known as the integral sign. The function $f(x)$ to be integrated is called the integrand and the expression dx informs us that the integration is performed with respect to x. If the integrand is a function of t instead, i.e. $f(t)$, and the integration is done with respect to t, we would have written dt instead. Finally, the indefinite integral of f is the family of antiderivatives given by $F(x) + c$, where c is known as the constant of integration.

Using this notation and the example above, we can thus write $\int 2x\,dx = x^2 + c$.

2.3 Basic rules of integration

Similar to differentiation, it is tedious if we had to make educated guesses at the antiderivatives before computing the indefinite integrals of functions. Accordingly, mathematicians have derived certain rules that will simplify the process of finding the indefinite integral. These rules are founded on the premise that differentiation and integration are reverse operations.

Rule 1: If k is a constant, then $\int k\,dx = kx + c$.

Observe that if we differentiate $f(x) = kx$, we would obtain the constant function $f'(x) = k$. In the "reverse" process, we would thus obtain the function $f(x) = kx$ by integrating the constant k.

Rule 2 (The Power Rule): If n is a real number, $n \neq -1$, then $\int x^n\,dx = \dfrac{1}{n+1}x^{n+1} + c$.

The power rule can be proved by differentiating the antiderivative on the right side of the formula. If the integrand involves a radical or a rational expression, we have to first rewrite the integrand using fractional or negative powers before using the Power Rule. An important point to note here is that the power rule does not work for $n = -1$ as this would mean dividing x^{n+1} by zero, which is indeterminate.

Rule 3: If k is a real number, then $\int kf(x)\,dx = k\int f(x)\,dx$.

The indefinite integral of a constant multiple of a function is equal to the constant multiple of the indefinite integral of the function. The result follows from the corresponding rule of differentiation. Note that k must be a constant and not another function of x.

Rule 4: If f and g are functions, then $\int \left[f(x) \pm g(x) \right] dx = \int f(x)\, dx \pm \int g(x)\, dx$.

The indefinite integral of a sum (or difference) of two functions is equal to the sum (or difference) of their indefinite integrals. This result follows from the corresponding rule of differentiation and it can be extended to the sum (or difference) of any finite number of functions.

Rule 5: (Exponential function) $\int e^x\, dx = e^x + c$.

The indefinite integral of the exponential function with base e is equal to the function itself. The rule does not hold for a general base b. It also does not hold if the exponent is a different function other than x.

Rule 6: $\int \frac{1}{x}\, dx = \int x^{-1}\, dx = \ln|x| + c, \quad x \neq 0$.

Recall that this case is the only exception to Rule 2 (The Power Rule) and the result follows directly from the derivative of ln $|x|$. The variable x is placed within a modulus sign as we can only take logarithms of a positive number. The rule does not hold if the denominator of the expression is a different function other than x. These basic rules of integration are summarised in Table 10.1.

Table 10.1: Basic rules of integration

Rule	Formula	Example
1.	$\int k\, dx = kx + c$	$\int 5\, dx = 5x + c$
2.	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$	$\int x^3\, dx = \frac{1}{3+1} x^{3+1} + c = \frac{1}{4} x^4 + c$
3.	$\int kf(x)\, dx = k \int f(x)\, dx$	$\int 2x^3\, dx = 2\int x^3\, dx = 2\left(\frac{1}{4}x^4\right) + c = \frac{1}{2}x^4 + c$
4.	$\int \left[f(x) \pm g(x) \right] dx$ $= \int f(x)\, dx \pm \int g(x)\, dx$	$\int x^3 + 5\, dx = \int x^3\, dx + \int 5\, dx = \frac{x^4}{4} + 5x + c$
5.	$\int e^x\, dx = e^x + c$	$\int 5e^x\, dx = 5\int e^x\, dx = 5e^x + c$
6.	$\int \frac{1}{x}\, dx = \int x^{-1}\, dx = \ln|x| + c, \quad x \neq 0$	$\int \frac{3}{x}\, dx = 3\int \frac{1}{x}\, dx = 3\ln|x| + c$

Example 10.1 Find the indefinite integral of the following.

(a) $\int 3+\sqrt{x}\,dx$ (b) $\int 2e^x-\frac{1}{x^2}\,dx$ (c) $\int \frac{1}{t}(t+1)\,dt$

Solution

(a) We rewrite the integrand in terms of fractional powers.

$$\begin{aligned}\int 3+\sqrt{x}\,dx &= \int 3+x^{\frac{1}{2}}\,dx \\ &= \int 3\,dx+\int x^{\frac{1}{2}}\,dx && \text{[Rule 4]} \\ &= 3x+\frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1}+c && \text{[Rule 1 and 2]} \\ &= 3x+\frac{2}{3}x^{\frac{3}{2}}+c\end{aligned}$$

(b) We rewrite the integrand in terms of negative powers, i.e.

$$\begin{aligned}\int 2e^x-\frac{1}{x^2}\,dx &= \int 2e^x-x^{-2}\,dx \\ &= \int 2e^x\,dx-\int x^{-2}\,dx && \text{[Rule 4]} \\ &=2\int e^x\,dx-\int x^{-2}\,dx && \text{[Rule 3]} \\ &= 2e^x-\frac{1}{-2+1}x^{-2+1}+c && \text{[Rule 2 and 5]} \\ &= 2e^x-\frac{1}{-1}x^{-1}+c \\ &= 2e^x+\frac{1}{x}+c\end{aligned}$$

(c) If possible, we simplify the integrand before integrating,

$$\begin{aligned}\int \frac{1}{t}(t+1)\,dt &= \int 1+\frac{1}{t}\,dt \\ &= \int 1\,dt+\int \frac{1}{t}\,dt && \text{[Rule 3]} \\ &= 1(t)+\ln|t|+c && \text{[Rule 1 and 6]} \\ &= t+\ln|t|+c\end{aligned}$$

3. Initial value problems

3.1 Differential equations

Suppose that we are given the rate of change of a function. We now have a technique to find the relation between the two quantities, i.e. integration. For example, if $f'(x) = 2x$, then $f(x)$ can be found by integrating the differentiated function to obtain $f(x) = x^2 + c$. Infinitely many functions have the same derivative $f'(x)$, with each differing from each other only by a constant.

In general, an equation that involves the derivative f' of an unknown function is called a differential equation. When we are asked to solve the differential equation, it means to find any function f that satisfies the equation. The solution with a constant c is known as a general solution as it gives all the solutions of f'. Finally, if we know the value of c due to a condition being stipulated on the function f, that solution is known as a particular solution.

Thus, in the example above, the differential equation is $f'(x) = 2x$. Solving the equation, the general solution is $f(x) = x^2 + c$. If an initial condition is specified, e.g. $f(2) = 5$, then we can obtain a value for c, e.g. $5 = 2^2 + c$ or $c = 1$ and the particular solution is $f(x) = x^2 + 1$. The specified condition that $f(2) = 5$ is an example of an initial condition, that is, a stipulated value of f for a certain value of x.

3.2 Initial value problems

An initial value problem is a problem in which we are required to find a particular solution of a differential equation. The problem will usually contain two elements: 1) a differential equation and 2) one or more initial conditions.

Example 10.2 Find the function f if it is known that $f'(x) = 6x^2 - 2x + 1$ and $f(1) = 4$.

Solution
Here, the differential equation is $f'(x) = 6x^2 - 2x + 1$ and the initial condition is $f(1) = 4$.
Integrating the function f',

$$\begin{aligned} f(x) &= \int 6x^2 - 2x + 1\,dx \\ &= 6\left(\frac{x^3}{3}\right) - 2\left(\frac{x^2}{2}\right) + 1(x) + c \\ &= 2x^3 - x^2 + x + c \end{aligned}$$

Using the initial condition,

$$\begin{aligned} f(1) &= 4 \\ 2(1)^3 - (1)^2 + (1) + c &= 4 \\ 2 + c &= 4 \\ c &= 2 \end{aligned}$$

Thus, the required function is

$$f(x) = 2x^3 - x^2 + x + 2$$

Example 10.3 The managing editor of a luxury car magazine wants to forecast the circulation of the magazine in 52 weeks. He projects a grown rate of

$$3+4t^{\frac{1}{3}}$$

copies per week, t weeks from now for the next 2 years. Given that the current circulation of the magazine is 2500 copies per week, what will be the projected circulation in 52 weeks?

Solution

If $C(t)$ denotes the relation between circulation and the number of weeks, then $C'(t)$ is the rate of change of circulation in the t^{th} week. So we have

$$C'(t)=3+4t^{\frac{1}{3}}$$

To obtain $C(t)$, we solve the differential equation.

$$\begin{aligned}C(t)&=\int 3+4t^{\frac{1}{3}}\ dt\\&=3t+\frac{4}{\frac{1}{3}+1}t^{\frac{1}{3}+1}+c\\&=3t+3t^{\frac{4}{3}}+c\end{aligned}$$

The current ($t = 0$) circulation of 2500 copies per week is translated into the initial condition, i.e. $C(0) = 2500$.

$$\begin{aligned}C(0)&=2500\\3(0)+3(0)^{\frac{4}{3}}+c&=2500\\c&=2500\end{aligned}$$

Thus, the current circulation is given by

$$C(t)=3t+3t^{\frac{4}{3}}+2500$$

The projected circulation in 52 weeks will be

$$\begin{aligned}C(52)&=3(52)+3(52)^{\frac{4}{3}}+2500\\&=3,238.27\\&\approx 3,238\end{aligned}$$

4. Integration by substitution

4.1 The method of substitution

The previous rules of integration dealt with simple functions and are closely correlated to the rules of differentiation. When we need to integrate functions that are more complicated, we will need a different technique. This section will introduce the method of substitution that is a powerful tool in integrating a large class of functions.

Essentially, the main aim of the method of substitution is to transform the integrand into a simple function in which we can use the basic rules of integration. To do so, we would have to substitute all the variables of x to a different dummy variable u as well as the component dx to du (to indicate that we are integrating the function with respect to u). Integration by substitution is derived from the chain rule in differentiation. However, the derivation will not be shown here.

There are 5 steps involved in integration by substitution:

1. Let $u = g(x)$, where $g(x)$ is part of the integrand, usually the "inside function" or a logarithm function.
2. Find $du = g'(x)\, dx$
3. Use the substitutions in steps 1 and 2 to convert the entire integral into one involving u.
4. Find the resulting integral. It should be possible to do so using the basic rules of integration.
5. Replace u by $g(x)$ to obtain the final solution in terms of x.

Example 10.4 Find the indefinite integral of the following.

(a) $\int 3x^2\left(x^3+1\right)^7\, dx$ (b) $\int 2\sqrt{2x-3}\, dx$ (c) $\int (2x+1)\left(x^2+x-1\right)^5\, dx$

Solution

(a) Step 1: Observe that the "inside function" is $x^3 + 1$.

$$\text{Let } u = x^3+1$$

Step 2: We "differentiate" the substitution, i.e. $du = g'(x)\, dx$

$$du = 3x^2\, dx$$

Step 3: Convert the entire integral into one involving u using Steps 1 and 2.

$$\int 3x^2\left(x^3+1\right)^7\, dx = \int \underbrace{\left(x^3+1\right)^7}_{u\text{ from1}} \underbrace{\left(3x^2dx\right)}_{du\text{ from2}}$$

$$= \int u^7\, du$$

Step 4: Find the integral using the basic rules.

$$\int u^7\, du = \frac{1}{8}u^8 + c$$

Step 5: Replace u by $x^3 + 1$ (Step 1) to obtain the final answer in terms of x.

$$\frac{1}{8}u^8 + c = \frac{1}{8}\left(x^3+1\right)^8 + c$$

(b) First, we rewrite the integrand in terms of indices.

$$\int 2\sqrt{2x-3}\ dx = \int 2(2x-3)^{\frac{1}{2}}\ dx$$

Step 1: Observe that the "inside function" is $2x - 3$.

$$\text{Let } u = 2x-3$$

Step 2: We "differentiate" the substitution, i.e. $du = g'(x)\ dx$

$$du = 2\ dx$$

Step 3: Convert the entire integral into one involving u using Steps 1 and 2.

$$\int 2(2x-3)^{\frac{1}{2}}\ dx = \int \underbrace{(2x-3)^{\frac{1}{2}}}_{u \text{ from1}} \underbrace{(2\ dx)}_{du \text{ from2}}$$

$$= \int u^{\frac{1}{2}}\ du$$

Step 4: Find the integral using the basic rules.

$$\int u^{\frac{1}{2}}\ du = \frac{2}{3}u^{\frac{3}{2}} + c$$

Step 5: Replace u by $2x - 3$ (Step 1) to obtain the final answer in terms of x.

$$\frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(2x-3)^{\frac{3}{2}} + c$$

(c) Step 1: Observe that the "inside function" is the function with an exponent.

$$\text{Let } u = x^2 + x - 1$$

Step 2: We "differentiate" the substitution, i.e. $du = g'(x)\ dx$

$$du = (2x+1)\ dx$$

Step 3: Convert the entire integral into one involving u using Steps 1 and 2.

$$\int (2x+1)(x^2+x-1)^5\ dx = \int \underbrace{(x^2+x-1)^5}_{u \text{ from1}} \underbrace{((2x+1)\ dx)}_{du \text{ from2}}$$

$$= \int u^5\ du$$

Step 4: Find the integral using the basic rules.

$$\int u^5\ du = \frac{1}{6}u^6 + c$$

Step 5: Replace u by $x^2 + x - 1$ (Step 1) to obtain the final answer in terms of x.

$$\frac{1}{6}u^6 + c = \frac{1}{6}(x^2+x-1)^6 + c$$

In the next example, the steps involved in integration will not be labelled.

Example 10.5 Find the indefinite integral of the following.

(a) $\displaystyle\int \frac{x}{2x^2-1}\ dx$ (b) $\displaystyle\int e^{-7x+1}\ dx$ (c) $\displaystyle\int \frac{(\ln x)^3}{2x}\ dx$

Solution

(a) The "inside function" is $2x^2 - 1$. We can think of the expression as $(2x^2 - 1)^{-1}$.

$$\text{Let } u = 2x^2 - 1$$
$$du = 4x\,dx$$

However, observe that we require $x\,dx$ only. So dividing both sides by 4 gives:

$$\frac{1}{4}du = x\,dx$$

Converting and integrating gives

$$\begin{aligned}\int \frac{x}{2x^2-1}\,dx &= \int \frac{1}{2x^2-1}\left(x\,dx\right)\\ &= \int \frac{1}{u}\left(\frac{1}{4}\,du\right)\\ &= \frac{1}{4}\int \frac{1}{u}\,du\\ &= \frac{1}{4}\ln|u| + c\\ &= \frac{1}{4}\ln\left|2x^2-1\right| + c\end{aligned}$$

(b) The "inside function" is $-7x + 1$.

$$\text{Let } u = -7x + 1$$
$$du = -7\,dx$$

However, observe that we require dx only. So dividing both sides by -7 gives:

$$-\frac{1}{7}du = dx$$

Converting and integrating gives

$$\begin{aligned}\int e^{-7x+1}\,dx &= \int e^{u}\left(-\frac{1}{7}du\right)\\ &= -\frac{1}{7}\int e^{u}\,du\\ &= -\frac{1}{7}e^{u} + c\\ &= -\frac{1}{7}e^{-7x+1} + c\end{aligned}$$

(c) As there is a logarithm in the integrand, the substitution should be the logarithm function.

$$\text{Let } u = \ln x$$
$$du = \frac{1}{x}\,dx$$

Converting and integrating gives

$$\begin{aligned}
\int \frac{(\ln x)^3}{2x}\,dx &= \frac{1}{2}\int \frac{(\ln x)^3}{x}\,dx \\
&= \frac{1}{2}\int (\ln x)^3 \left(\frac{1}{x}\,dx\right) \\
&= \frac{1}{2}\int u^3\,du \\
&= \frac{1}{2}\left[\frac{u^4}{4}\right] + c \\
&= \frac{1}{8}u^4 + c \\
&= \frac{1}{8}(\ln x)^4 + c
\end{aligned}$$

The method of substitution can be used in initial value problems as well. The main difference is that we would use integration by substitution to solve the differential equation.

Example 10.6 The senior head of research in a firm manufacturing microchips predicts that the cost of producing microchips would drop at the rate of

$$\frac{72}{(2t+3)^2}, \qquad (0 \le t \le 10)$$

dollars for the next t years. In the year 2017 (t = 0), each microchip costs \$12. What is his prediction for the cost of a microchip in the year 2027 (t = 10)?

Solution

If $C(t)$ denotes the relation between cost of a microchip and the time in years, then $C'(t)$ is the rate of change of cost in the t^{th} year. So we have

$$C'(t) = -\frac{72}{(2t+3)^2} = -72(2t+3)^{-2}$$

To obtain $C(t)$, we solve the differential equation.

$$C(t) = -\int 72(2t+3)^{-2}\,dt$$

As we are unable to simplify the integrand algebraically, we use integration by substitution.

$$\text{Let } u = 2t + 3$$
$$du = 2\,dt$$

$$\begin{aligned}
C(t) &= -\int 72(2t+3)^{-2}\,dt \\
&= -\int 72u^{-2}\left(\frac{1}{2}du\right) \\
&= -36\int u^{-2}\,du \\
&= -36\left(\frac{1}{-1}\right)u^{-1} + c \\
&= \frac{36}{u} + c \\
&= \frac{36}{2t+3} + c
\end{aligned}$$

The current cost is \$10 for each cell, so the initial condition is $C(0) = 12$.

$$\begin{aligned}
C(0) &= 12 \\
\frac{36}{2(0)+3} + c &= 12 \\
c &= 0
\end{aligned}$$

Thus, the current cost is given by

$$C(t) = \frac{36}{2t+3}$$

The predicted cost in 10 years will be

$$\begin{aligned}
C(10) &= \frac{36}{2(10)+3} \\
&\approx \$1.57
\end{aligned}$$

5. Discussion questions

1. Find the indefinite integral of the following.

(a) $\int x^{10}\,dx$ (b) $\int 7-\sqrt{x}\,dx$

(c) $\int \frac{8}{x^3}+\frac{1}{\sqrt{x}}\,dx$ (d) $\int x(x+1)\,dx$

(e) $\int 4e^x-3\,dx$ (f) $\int \frac{1}{x^2}(x-1)\,dx$

2. Find the function f if it is known that $f'(x) = 2x^3 + x - 5$ and $f(2) = 7$.

3. The average credit card debt per household in a particular country was growing at the rate of approximately $-4.02t^2 + 60t + 240$ dollars per year between the year 2000 ($t = 0$) and 2015 ($t = 15$). It was found that the average credit card debt in 2000 was \$3000.

(a) Find an expression giving the approximate average credit card debt per household in year t where $0 \le t \le 15$.

(b) Estimate the average credit card debt per household in 2009.

4. Pauline's Phone Pte Ltd manufactures telephones. The daily marginal cost for producing x units of telephones is given by the function

$$C'(x) = 0.000012x^2 - 0.008x + 10$$

where $C'(x)$ is measured in dollars per unit. The daily fixed cost incurred in producing these telephones is \$150. Find the total cost incurred for producing 200 telephones.

5. Find the indefinite integral of the following.

(a) $\int 6x^2\left(2x^3+3\right)^4\,dx$ (b) $\int 4x\sqrt{2x^2-1}\,dx$

(c) $\int \frac{2x}{3x^2-1}\,dx$ (d) $\int \frac{2e^x}{1+e^x}\,dx$

(e) $\int e^{0.01x}\,dx$ (e) $\int \frac{\ln 5x}{x}\,dx$

6. The viewership of a reality TV show has been increasing at the rate of

$$3\left(5+\frac{1}{2}t\right)^{-\frac{1}{3}} \quad (0 \le t \le 5)$$

million viewers per year where $t = 0$ corresponds to the year 2015. Given that the number of viewers in 2015 is approximately 30 million, how many viewers are expected in the year 2020?

6. Supplementary questions

1. Find the indefinite integral of the following.

(a) $\int 2e^x - \frac{1}{x}\,dx$ (b) $\int (x+1)^2\,dx$

(c) $\int (x+7)(x-7)\,dx$ (d) $\int \frac{5x^2 - x + 3}{x}\,dx$

2. Find the function f if it is known that $f'(x) = e^x + x - 2$ and $f(0) = 3$.

3. A study conducted by a new network channel estimates that the number of subscribers for its network channel will grow at the rate of

$$150 + 238t^{\frac{3}{4}}$$

new subscribers per month, t months from the start date of its service. If 2000 subscribers signed up for the service before the starting date, how many subscribers will there be 10 months later?

4. The estimated marginal profit per month associated with producing and selling scientific calculators is $P'(x) = -0.004x + 20$, where $P'(x)$ is measured in dollars per unit when the production level is x units. It is given that the monthly fixed cost is \$10,000.

(a) At what level of production is the maximum profit realized?

(b) What is the maximum monthly profit?

5. Find the indefinite integral of the following.

(a) $\int (3x^2 - 2)(x^3 - 2x)^3\,dx$ (b) $\int e^{2-x}\,dx$

(c) $\int 2xe^{x^2}\,dx$ (d) $\int \frac{2}{x\ln x}\,dx$

6. The average student in a 10-week speed typing course at a training institute progresses according to the rule

$$N'(t) = 6e^{-0.04t} \qquad (0 \le t \le 10)$$

where $N'(t)$ measures the rate of change in the number of words the student types per minute after t weeks of the course. Given that the average student starts the course typing at the speed of 50 words per minute, what is the estimated typing speed of the student when he graduates from the course?

BUSINESS MATHEMATICS

SESSION 11

INTEGRATION AND ITS APPLICATION II

At the end of the session, students should be able to:

1. Relate area of regions enclosed by graphs to definite integrals.
2. Know and apply the Fundamental Theorem of Calculus.
3. Apply definite integrals to measure net change of a quantity and to compute the average value of a function.

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1. Introduction

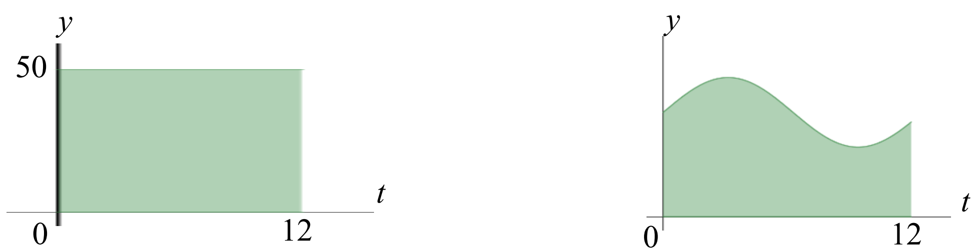
This chapter follows from the previous chapter in exploring the second fundamental problem of calculus, that is finding the area of region under a graph. To compute this requires us to introduce the new concept of a definite integral. In turn, the definite integral allows us to establish a link between differential and integral calculus via the Fundamental Theorem of Calculus. Some of the applications of the theorem are discussed, i.e. to measure how much a quantity has changed over time and to compute the average value of a function.

2. Area of regions enclosed by graphs

2.1 Area under a graph

Suppose that the water consumption each month in a particular household is constant and given by the function $f(t) = 50$, where t is measured in months and $f(t)$ in cubic meters per month. Over a period of 12 months, the total water consumption is given by 50 x 12 = 600 cubic meters. This information can be illustrated by the graph of f shown in Figure 11.1(a).

Figure 11.1: Total water consumption over 12 months



(a) Total water consumption is given by area of shaded region = 50 x 12 = 600 cubic meters

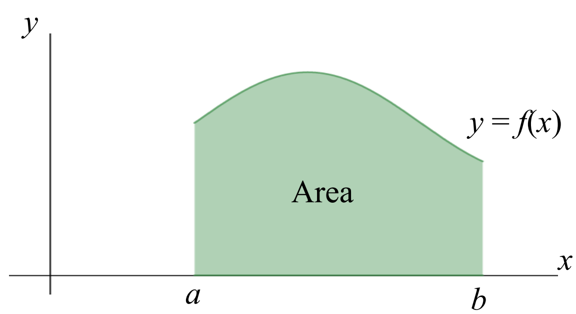
(b) Total water consumption is given by area of shaded region as well.

In reality, the water consumption each month will not be constant. However, we can still present the information in a graph as shown in Figure 11.1(b). Intuitively, the total water consumption over a period of 12 months can be computed by the shaded area as well. This brings us to the second fundamental problem of calculus: Given the graph of a function, how would we compute the area enclosed by it?

2.2 The Riemann sum

We first define the specific area that we wish to compute. Given a non-negative function f, the area of the region under f on the interval $[a, b]$ is defined as the area enclosed by f, the x-axis and the vertical lines $x = a$ and $x = b$ (See Figure 11.2).

Figure 11.2: The area of region under f



The essential idea to compute the area under a curve is to first approximate the area by using a number of rectangles with the same width. Subsequently, the number of rectangles is increased to an infinite number and we can thus equate the two areas. A formula is then defined to compute the area of the rectangles. We will use a specific example to demonstrate this in action.

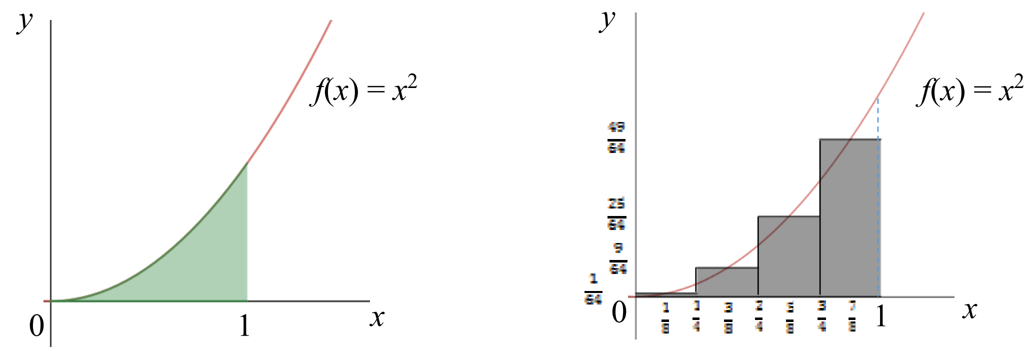
Example 11.1 Estimate the area of region R under the graph of $f(x) = x^2$ on the interval $[0, 1]$ using rectangles with (a) 4 subintervals, (b) 8 subintervals, (c) 16 subintervals.

Solution
(a) The region that is required is shown in Figure 11.3(a).

To obtain an approximation of R with 4 subintervals, we first divide the interval $[0, 1]$ into 4 equal subintervals, each with length $\frac{1}{4}$. The subintervals are thus $\left[0,\frac{1}{4}\right],\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{1}{2},\frac{3}{4}\right],\left[\frac{3}{4},1\right]$.

Next we construct rectangles with these subintervals as bases. The height of each rectangle is given by the value of the function at the midpoints of the subintervals. The mid points of the subintervals are $\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$. Thus, the corresponding heights are $f\left(\frac{1}{8}\right),f\left(\frac{3}{8}\right),f\left(\frac{5}{8}\right),f\left(\frac{7}{8}\right)$ or $\frac{1}{64},\frac{9}{64},\frac{25}{64},\frac{49}{64}$. This approximation is shown in Figure 11.3(b)

Figure 11.3: Area of region R with 4 subintervals



(a) The area of region R that is required.

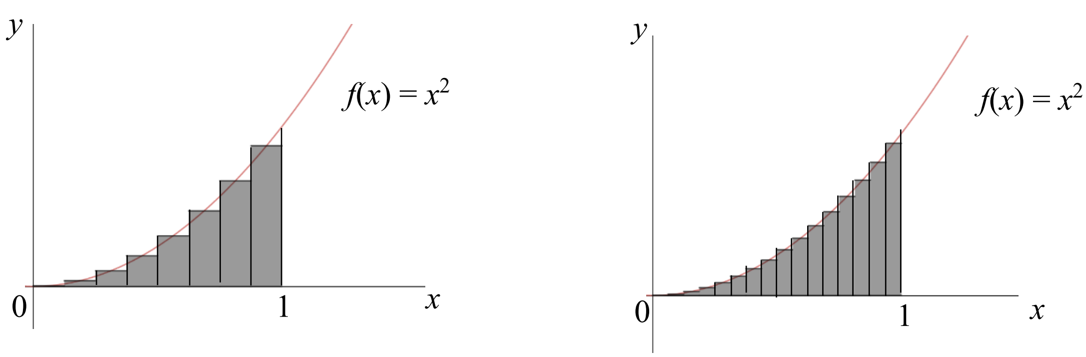
(b) An approximation of the area of region R with 4 subintervals.

If the area R is approximated by the area of the four rectangles, then we have

$$\begin{aligned}
R &\approx \frac{1}{4}f\left(\frac{1}{8}\right)+\frac{1}{4}f\left(\frac{3}{8}\right)+\frac{1}{4}f\left(\frac{5}{8}\right)+\frac{1}{4}f\left(\frac{7}{8}\right) \\
&= \frac{1}{4}\left[f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right] \\
&= \frac{1}{4}\left[\frac{1}{64}+\frac{9}{64}+\frac{25}{64}+\frac{49}{64}\right] \\
&= \frac{21}{64}
\end{aligned}$$

(b), (c) Following the same procedure as in (a), we can approximate the area of region R using 8 rectangles and 16 rectangles, as illustrated in Figure 11.4(a) and Figure 11.4 (b).

Figure 11.4: Area of region R with 4 subintervals



(a) An approximation of the area of region R with 8 subintervals.

(b) An approximation of the area of region R with 16 subintervals.

Figures 11.3 and 11.4 seem to indicate that the approximations get better as the number of rectangles increases. If we do a computation for (b) and (c), we obtain the following.

When the area R is approximated by the area of 8 rectangles, we have

$$R \approx \frac{1}{4}\left[f\left(\frac{1}{16}\right)+f\left(\frac{3}{16}\right)+f\left(\frac{5}{16}\right)+f\left(\frac{7}{16}\right)+f\left(\frac{9}{16}\right)+f\left(\frac{11}{16}\right)+f\left(\frac{13}{16}\right)+f\left(\frac{15}{16}\right)\right]=0.332031$$

When the area R is approximated by the area of 16 rectangles, we have

$$R \approx \frac{1}{4}\left[f\left(\frac{1}{32}\right)+f\left(\frac{3}{32}\right)+f\left(\frac{5}{32}\right)+f\left(\frac{7}{32}\right)+\ldots+f\left(\frac{31}{32}\right)\right]=0.333008$$

As the number of rectangles continue to increase, we obtain the following approximations.

Number of rectangles	4	8	16	32	64	100	200
Approximation of R	0.32813	0.33203	0.33301	0.33325	0.33331	0.33333	0.33333

The computations appear to approach the number $\frac{1}{3}$. Therefore, we define the area of the region under the graph of $f(x) = x^2$ in the interval [0, 1] to be $\frac{1}{3}$.

The example above can be generalised to approximate and define the area of the region under the graph of a continuous non-negative function f in any interval $[a, b]$. To approximate the area using n number of rectangles, we perform the following.

1) Divide the interval $[a, b]$ into n subintervals of equal length, $\Delta x = \frac{b-a}{n}$. This is the length of each rectangle.
2) Compute the midpoint of each subinterval, denoted by $x_1, x_2, \ldots, x_n$.
3) Compute the value of f at the midpoints found in 2), i.e. $f(x_1), f(x_2), \ldots, f(x_n)$. These will be the heights of each rectangle.
4) The area of region R is approximated by: $R \approx \Delta x\left[f(x_1)+f(x_2)+\ldots+f(x_n)\right]$. This is known as the Riemann sum, in honour of the German mathematician Bernhard Riemann (1826 – 1866).

The area of the region R is then defined by $R = \lim_{n\to\infty} \Delta x\left[f(x_1)+f(x_2)+\ldots+f(x_n)\right]$.

2.3 The definite integral

The limit of the Riemann sum gives the area under the graph of a continuous non-negative function on an interval $[a, b]$. However, many functions are not necessarily non-negative. To compute the area under the graph of these functions, we will start with the following definition.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x\left[f(x_1)+f(x_2)+\ldots+f(x_n)\right]$$

The notation $\int_a^b f(x)\,dx$ is called the definite integral of f from a to b. The number a is known as the lower limit of integration and the number b is known as the upper limit of integration.

Note that the definite integral is a number whereas the indefinite integral $\int f(x)\,dx$ gives a family of functions. To evaluate the definite integral without calculating the various Riemann sums, we would have to make use of the fundamental theorem of calculus.

3. The fundamental theorem of calculus

3.1 The fundamental theorem of calculus

Let f be a continuous function on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b)-F(a), \quad \text{where } F'(x)=f(x).$$

When evaluating the definite integral, we will use the notation $\left[F(x)\right]_a^b$ to denote $F(b)-F(a)$.

Example 11.2 Evaluate the following definite integrals: (a) $\int_1^2 4x^3\,dx$, (b) $\int_1^3 \frac{1}{x}+5\,dx$

Solution

(a) We integrate the function before substituting the values of b and a.

$$\begin{aligned}\int_1^2 4x^3\,dx &= \left[4\left(\frac{x^4}{4}\right)\right]_1^2 && \text{[Integrate the function]}\\ &= \left[x^4\right]_1^2 \\ &= 2^4-1^4 && [F(b)-F(a)]\\ &= 15\end{aligned}$$

(b) The process is similar for all functions.

$$\begin{aligned}\int_1^3 \frac{1}{x}+5\,dx &= \left[\ln x+5x\right]_1^3\\ &= \left[\ln 3+5(3)\right]-\left[\ln 1+5(1)\right]\\ &= \ln 3+10\\ &\approx 11.1\end{aligned}$$

3.2 Area of region under graph

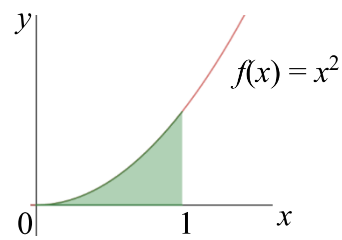
The fundamental theorem of calculus can be used to compute the area of region under a curve. The reason for this can be found in the appendix at the end of this chapter.

Example 11.3 Find the area of region R under the graph of $f(x) = x^2$ on the interval [0, 1]

Solution

In Example 11.1, we defined the area to be $\frac{1}{3}$. We will verify this result here.

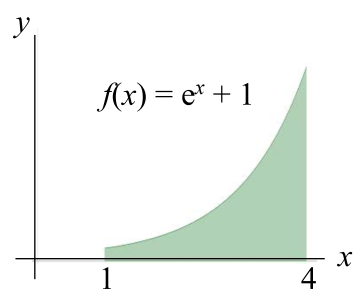
$$\begin{aligned}\int_0^1 x^2\,dx &= \left[\frac{x^3}{3}\right]_0^1 \\ &= \frac{1^3}{3} - \frac{0^3}{3} \\ &= \frac{1}{3}\end{aligned}$$



Example 11.4 Find the area of region R under the graph of $f(x) = e^x + 1$ on the interval [1, 4]

Solution

$$\begin{aligned}\text{Area of region} &= \int_1^4 e^x + 1\,dx \\ &= \left[e^x + x\right]_1^4 \\ &= \left[e^4 + 4\right] - \left[e^1 + 1\right] \\ &\approx 54.9\end{aligned}$$



3.3 Properties of the definite integral

Listed in Table 11.1 below are some properties of the definite integral, which can be derived from the fundamental theorem of calculus. The proofs will not be shown here.

Table 11.1: Properties of the definite integral

No.	Property
1.	$\int_a^a f(x)\,dx = 0$
2.	$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3.	$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is a constant
4.	$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5.	$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

3.4 The method of substitution for definite integrals

When we need to find the definite integral of functions that are more complicated, we can use the method of substitution to aid us. There are two approaches that are generally applied and we will examine these approaches in the next example.

Example 11.5 Evaluate $\int_0^3 2x\left(16+x^2\right)^{\frac{1}{2}}\,dx$.

Solution

Method 1: First find the corresponding indefinite integral $\int 2x\left(16+x^2\right)^{\frac{1}{2}}\,dx$ before substituting in the limits.

$$\text{Let } u = 16 + x^2$$
$$du = 2x\,dx$$
$$\int 2x\left(16+x^2\right)^{\frac{1}{2}}\,dx = \int u^{\frac{1}{2}}\,du$$
$$= \frac{2u^{\frac{3}{2}}}{3} + c$$
$$= \frac{2\left(16+x^2\right)^{\frac{3}{2}}}{3} + c$$

Using this result and substituting the limits:

$$\int_0^3 2x\left(16+x^2\right)^{\frac{1}{2}}\,dx = \left[\frac{2\left(16+x^2\right)^{\frac{3}{2}}}{3}\right]_0^3$$
$$= \left[\frac{2\left(16+3^2\right)^{\frac{3}{2}}}{3}\right] - \left[\frac{2\left(16+0^2\right)^{\frac{3}{2}}}{3}\right]$$
$$= \frac{250}{3} - \frac{128}{3}$$
$$= \frac{122}{3}$$

Method 2: Note that the numbers 0 and 3 are the lower and upper limit of x-values. The second approach requires us to change the limits of integration to u-values.

The first two steps are the same.

$$\text{Let } u = 16 + x^2$$
$$du = 2x\,dx$$

Since we are converting the integral into one involving only u, we will perform a substitution for the limits as well, using the first step.

$$\text{When } x = 0,\ u = 16 + 0^2 = 16$$
$$\text{When } x = 3,\ u = 16 + 3^2 = 25$$

Therefore,

$$\begin{aligned}\int_{x=0}^{x=3} 2x\left(16+x^2\right)^{\frac{1}{2}}\,dx &= \int_{u=16}^{u=25} u^{\frac{1}{2}}\,du \\ &= \left[\frac{2u^{\frac{3}{2}}}{3}\right]_{16}^{25} \\ &= \frac{2(25)^{\frac{3}{2}}}{3} - \frac{2(16)^{\frac{3}{2}}}{3} \\ &= \frac{122}{3}\end{aligned}$$

4. Other applications of the definite integral

4.1 The net change of a quantity

In some real world applications, we are interested in the net change of a quantity over a period of time. For example, we might want to compute how much a population has changed over ten years or how much the production costs changes as the quantity produced changes. If we can represent the rate of change as a function, we can calculate the net change of a quantity by evaluating an appropriate definite integral. Thus, the net change of a function f over an interval $[a, b]$ is given by

$$f(b) - f(a) = \int_a^b f'(x)\,dx$$

Example 11.6 The daily marginal cost function associated with producing a line of children's toy train is given by the function

$$C'(x) = 0.000006x^2 - 0.006x + 3.$$

Find the daily total cost required to produce the 101^{st} to 200^{th} unit.

Solution

Variable cost for producing 101st to 200th unit $= C(200) - C(100) = \int_{100}^{200} C'(x)\,dx$

$$= \int_{100}^{200} 0.000006x^2 - 0.006x + 3\,dx$$

$$= \Big[0.000002x^3 - 0.003x^2 + 3x\Big]_{100}^{200}$$

$$= \Big[0.000002(200)^3 - 0.003(200)^2 + 3(200)\Big]$$

$$- \Big[0.000002(100)^3 - 0.003(100)^2 + 3(100)\Big]$$

$$= 224 \quad \text{or} \quad \$224$$

4.2 The average value of a function

Recall that the average value of a set of numbers is obtained by adding up the values and dividing by the number of values in the set, e.g. the average value of 1, 3 and 8 is computed by (1+3+8)/3 which gives the answer 4.

Now suppose that we want to obtain the average value of a function over an interval $[a, b]$. We can approximate this value by dividing the interval into n subintervals of equal length (similar to Section 2.2), choosing the x-coordinates at each subinterval, x_1, x_2, …, x_n and computing the value of the function at each point, i.e. $f(x_1)$, $f(x_2)$, …, $f(x_n)$. The average value of the function is thus approximated by $\dfrac{f(x_1) + f(x_2) + ... + f(x_n)}{n}$.

Manipulating the expression, we obtain

$$\frac{f(x_1) + f(x_2) + ... + f(x_n)}{n} = \frac{1}{b-a}\left[\frac{b-a}{n} f(x_1) + \frac{b-a}{n} f(x_2) + ... + \frac{b-a}{n} f(x_n)\right]$$

$$= \frac{1}{b-a}\Big[(\Delta x) f(x_1) + (\Delta x) f(x_2) + ... + (\Delta x) f(x_n)\Big]$$

$$= \frac{1}{b-a}\Delta x\Big[f(x_1) + f(x_2) + ... + f(x_n)\Big]$$

From Section 2.2, we know that as n increases, $\lim_{n\to\infty} \Delta x\Big[f(x_1) + f(x_2) + ... + f(x_n)\Big] = \int_a^b f(x)\,dx$.
Therefore, we can say that:

$$\text{Average value of } f \text{ over the interval } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example 11.7 Find the average value of the function $f(x) = 2x^4 + 1$ over the interval [0, 3].

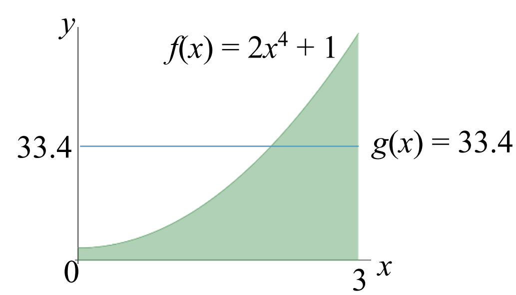
Solution

$$\begin{aligned}
\text{Average value} &= \frac{1}{3-0}\int_0^3 2x^4 + 1\,dx \\
&= \frac{1}{3}\left[\frac{2x^5}{5} + x\right]_0^3 \\
&= \frac{1}{3}\left[\left(\frac{2(3)^5}{5} + 3\right) - \left(\frac{2(0)^5}{5} + 0\right)\right]_0^3 \\
&= \frac{501}{15} \quad \text{or} \quad 33.4
\end{aligned}$$

The average value of a function is the mean of all the values of $f(x)$ in the interval $[a, b]$. Geometrically, this means that we can replace the function over the interval with a constant function $g(x) = k$, where k is the average value, and the area under the two functions will be the same.

In Example 11.7, we can compute the area under the graph.

$$\begin{aligned}
\text{Area of region under graph} &= \int_0^3 2x^4 + 1\,dx \\
&= \left[\frac{2x^5}{5} + x\right]_0^3 \\
&= \left[\left(\frac{2(3)^5}{5} + 3\right) - \left(\frac{2(0)^5}{5} + 0\right)\right]_0^3 \\
&= 100.2
\end{aligned}$$



If we replace $f(x)$ by the function $g(x) = 33.4$, (33.4 is the average value as computed previously),

$$\begin{aligned}
\text{Area of region under graph} &= \text{Area of rectangle} \\
&= 33.4 \times 3 \\
&= 100.2
\end{aligned}$$

Thus, the average value of a function f in an interval $[a, b]$ can also be defined as the height of a rectangle with a base of length $(b - a)$ that has the same area as the region under the graph of f in the same interval $[a, b]$.

5. Discussion questions

1. Let $f(x) = x^2 + 1$ and compute the Riemann sum of f over the interval [2, 4] using (a) 2 subintervals of equal length and (b) 5 subintervals of equal length.

2. Evaluate the following definite integrals.

(a) $\int_3^{12} -2\,dx$ (b) $\int_1^4 \sqrt{x} - \frac{1}{x^2}\,dx$

(c) $\int_1^5 x\left(\frac{4}{x^2} - \frac{1}{x}\right)dx$ (d) $\int_0^3 e^x - x\,dx$

3. Find the area under the graph of the function $f(x) = (x^2 - 1)^2$ in the interval [1, 3].

4. Evaluate the following definite integrals.

(a) $\int_0^1 5(5x-1)^7\,dx$ (b) $\int_0^2 \frac{1}{\sqrt{2x+1}}\,dx$

(c) $\int_1^2 2xe^{x^2}\,dx$ (d) $\int_1^4 \frac{1}{3x-1}\,dx$

5. Find the area of region under the graph of the function $f(x) = 3x^2\left(x^3+1\right)^{\frac{1}{2}}$ in the interval [0, 1].

6. The annual sales (in millions of units) of Krystal tablet PC are expected grow according to the function $f(t) = 0.18t^2 + 0.16t + 2.64$, $(0 \le t \le 6)$ per year, where t is measured in years. How many tablet PCs were sold in the next 6 years?

7. Find the average value of the functions f over the interval $[a, b]$.

(a) $\int 2x^2 - 5\,dx$; [1, 5] (b) $\int \frac{1}{2x+3}\,dx$; [0, 3]

8. The total petrol consumption in a particular country can approximated by the function

$$P(t) = 0.014t^2 + 1.93t + 130, \quad (0 \le t \le 10)$$

where $t = 0$ corresponds to the end of the year 2005, $t = 10$ corresponds to the end of the year 2015 and $P(t)$ is measured in millions of litres per year.

(a) What is the total petrol consumption at the end of the year 2014?

(b) What is the average consumption per year over the period from the end of 2005 to the end of 2015?

6. Supplementary questions

1. Let $f(x) = e^x$ and compute the Riemann sum of f over the interval [0, 2] using (a) 2 subintervals of equal length and (b) 5 subintervals of equal length.

2. Evaluate the following definite integrals.

(a) $\int_{-5}^{-3} 1\,dx$ (b) $\int_{1}^{9} x+\frac{1}{\sqrt{x}}\,dx$

(c) $\int_{1}^{2} \frac{x^3+4x-1}{x}\,dx$ (d) $\int_{-1}^{1} 7x-e^x\,dx$

3(a) Sketch the graph of the function $f(x) = e^x + 1$.

(b) Find the area of region under the graph of the function $f(x) = e^x + 1$ in the interval [0, 5].

4. Evaluate the following definite integrals.

(a) $\int_{0}^{1} 14x\left(7x^2-1\right)^3\,dx$ (b) $\int_{2}^{7} \frac{1}{3\sqrt{x+2}}\,dx$

(c) $\int_{0}^{2} xe^{x^2+1}\,dx$ (d) $\int_{2}^{3} \frac{2x}{x^2-1}\,dx$

5. Find the area under the graph of the function $f\left(x\right)=x\left(x^2-6\right)^{\frac{1}{2}}$ in the interval [7, 10].

6. A division of Sam's Electronics manufactures rice cookers. The daily marginal cost of producing these rice cookers are given by the function

$$C'\left(x\right)=0.0003x^2-0.12x+20$$

where x is the number of units produced and $C'(x)$ is measured in dollars per unit. The daily fixed cost incurred is \$1000.

(a) Find the total cost incurred in producing the first 400 units per day.

(b) What is the cost incurred in producing the 201st to 300th unit?

7. Find the average value of the functions f over the interval $[a, b]$.

(a) $\int x^3+x\,dx$; [0, 2] (b) $\int e^{3x}\,dx$; [1, 2]

8. In a certain country, the number of knee cap replacements procedures (in millions) performed from 2000 to 2040 is projected to be

$$f\left(t\right)=0.0000848t^3+0.002112t^2+0.03898t+0.16,\quad \left(0\le t\le 40\right)$$

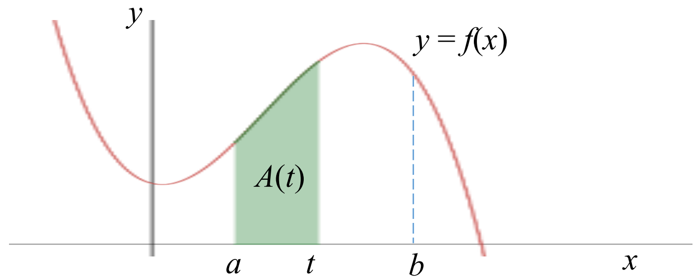
where $t = 0$ corresponds to the year 2000.

(a) What is the total number of knee cap replacement procedures in 2000?

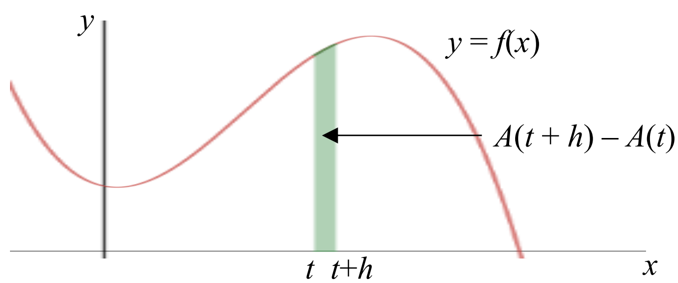
(b) What is the projected average number of total knee cap replacement procedures performed from 2000 to 2040?

Appendix: Using the Fundamental Theorem of Calculus to compute the area of region under graph

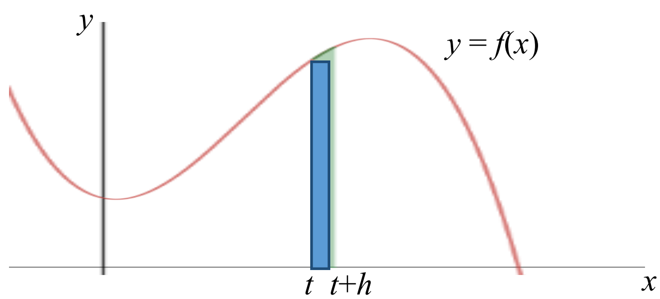
1. Define an "area function" $A(t)$ which represents the area of the region R under the graph of $y = f(x)$ from $x = a$ to $x = t$, where $a \leq t \leq b$.



2. Let h be a small positive number. Then $A(t + h)$ is the area of the region under the graph of $y = f(x)$ from $x = a$ to $x = t + h$. Observe that by taking the difference, $A(t + h) - A(t)$ is the area of the region under the graph of $y = f(x)$ from $x = t$ to $x = t + h$.



3. The difference in areas can be approximated by a rectangle with width h and height $f(t)$.



4. Thus we have

$$A(t+h) - A(t) \approx h \times f(t)$$

$$\frac{A(t+h) - A(t)}{h} \approx f(t)$$

5. By the definition of the derivative, $\lim_{h \to 0} \frac{A(t+h) - A(t)}{h} = A'(t)$. Since $f(t)$ does not depend on h at all and because the approximation becomes exact when h approaches zero, we have

$$A'(t) = \lim_{h \to 0} \frac{A(t+h) - A(t)}{h} = f(t)$$

6. We have shown that the area function $A(t)$ is an antiderivative of the function f since the equation in 5 holds for all values of t in the interval $[a, b]$. Hence by integration, the area of the region of graph under f must be of the form $F(x) + c$, where $F(x)$ is any antiderivative of f and c is a constant.

7. Therefore we have the relationship $A(x) = F(x) + c$. To find the area of region under the graph over the interval $[a, b]$, we make two observations.

   (i) $A(a) = 0 \Rightarrow F(a) + c = 0 \Rightarrow c = -F(a)$

   (ii) Area of region = $A(b) \Rightarrow$ Area of region = $F(b) + c \Rightarrow$ Area of region = $F(b) - F(a)$

8. Finally since the area of the region under the graph is $\int_a^b f(x)\,dx$, we have

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

BUSINESS MATHEMATICS

SESSION 12

INTEGRATION AND ITS APPLICATION III

At the end of the session, students should be able to:

1. Compute areas between two curves.
2. Apply the areas between two curves to compute consumers' and producers' surplus.
3. Apply definite integrals to compute the future and present value of an income stream and future and present value of an annuity.

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1. Introduction

This chapter follows from the previous chapter in exploring further applications of the definite integral in business and economics. We will discuss a technique to compute the area of region between two curves, which in turn will lead us to calculate the consumers' and producers' surplus. Additionally, the definite integral will aid us in computing the future and present value of an income stream as well as the future and present value of an annuity.

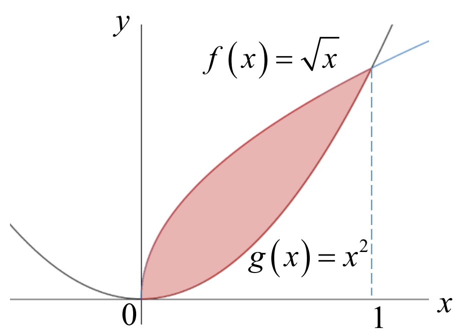
2. Area of region between two graphs

2.1 Area between two graphs

Recall from the previous chapter that the area under the graph of a function *f* on an interval [a ,b] is given by the definite integral $\int_a^b f(x)\,dx$. Using this definition, we can solve the general problem of evaluating the area between the graphs of two different functions, *f* and *g*.

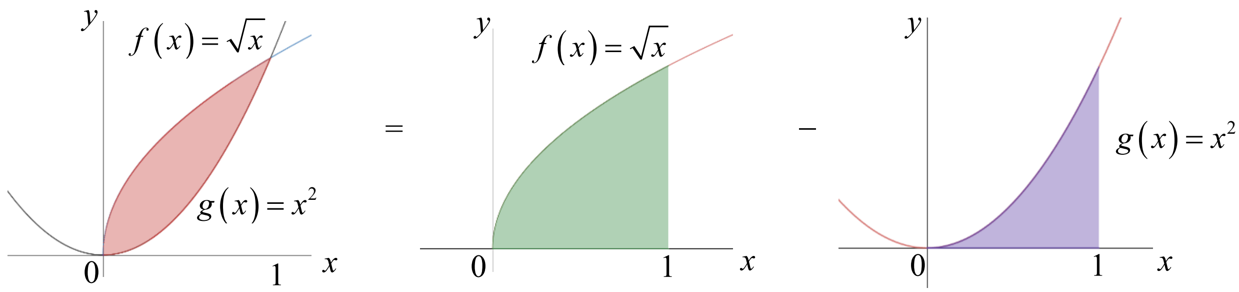
First, consider a specific case where $f(x)=\sqrt{x}$, $g(x)=x^2$ and we want to find the area between these two functions on the interval [0, 1]. This area is illustrated in Figure 12.1.

Figure 12.1: Area between the functions $f(x)=\sqrt{x}$ and $g(x)=x^2$ on the interval [0, 1]



Using definite integrals, we can calculate the area under the graph of f and the area under the graph of g. Observe that the required area between the two graphs can be evaluated by computing the difference between these two areas, as shown in Figure 12.2.

Figure 12.2: Required area is the difference between two areas under graphs



$$\text{Area between the two graphs} = \int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx$$

$$= \int_0^1 \sqrt{x} - x^2\,dx \qquad \text{[Using a property of the definite integral]}$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{1^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1^3}{3}\right] - \left[\frac{0^{\frac{3}{2}}}{\frac{3}{2}} - \frac{0^3}{3}\right]$$

$$= \frac{1}{3}$$

Generalizing the above example, we obtain the following result. If f and g are continuous functions such that $f(x) \geq g(x)$ on the interval $[a, b]$, then the area of the region between the graphs of f and g on $[a, b]$ is given by:

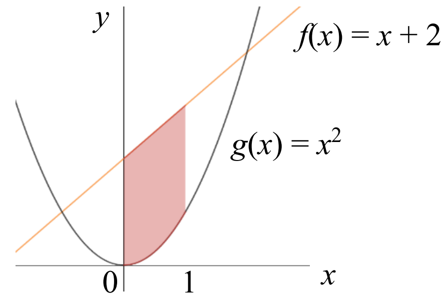
$$\text{Area between two graphs} = \int_a^b \left[f(x) - g(x)\right]dx$$

Informally, we evaluate the integral of the “upper” curve minus the “lower” curve.

Example 12.1 Find the area of the region R bounded by the graphs of $f(x) = x + 2$, $g(x) = x^2$ and the vertical lines $x = 0$ and $x = 1$.

Solution
The required area is shown below. In this case, the “upper” curve is $f(x) = x + 2$, the lower curve is $g(x) = x^2$, the lower limit is 0 and the upper limit is 1.

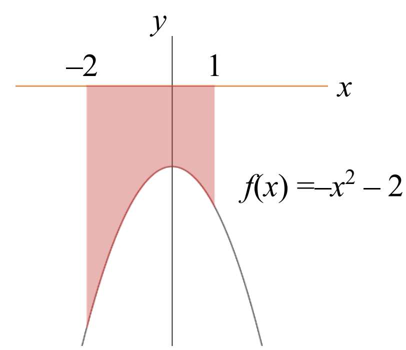
$$\text{Required area} = \int_0^1 (x+2)-(x^2)\,dx$$
$$= \int_0^1 x+2-x^2\,dx$$
$$= \left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_0^1$$
$$= \frac{13}{6}\ \text{unit}^2$$



Example 12.2 Find the area of the region R bounded by the graphs of $f(x) = -x^2 - 2$, the x-axis and the vertical lines $x = -2$ and $x = 1$.

Solution
The required area is shown below. In this case, the "upper" curve is the x-axis which is also defined by the function $f(x) = 0$, the lower curve is $f(x) = -x^2 - 2$, the lower limit is –2 and the upper limit is 1.

$$\text{Required area} = \int_{-2}^1 (0)-(-x^2-2)\,dx$$
$$= \int_{-2}^1 x^2+2\,dx$$
$$= \left[\frac{x^3}{3}+2x\right]_{-2}^1$$
$$= \left[\frac{1^3}{3}+2(1)\right]-\left[\frac{(-2)^3}{3}+2(-2)\right]$$
$$= \frac{28}{3}\ \text{unit}^2$$

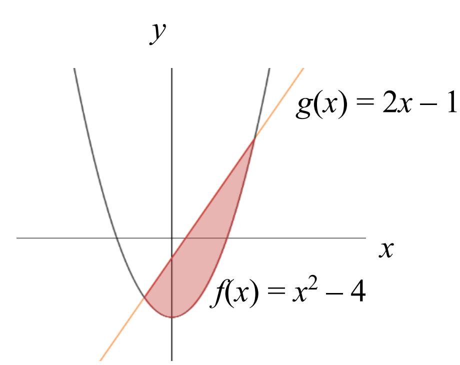


Example 12.3 Find the area of the region that is completely enclosed by the graphs of the functions $f(x) = x^2 - 4$ and $g(x) = 2x - 1$.

Solution
The required area is shown below. In this case, the "upper" curve is $g(x) = 2x - 1$, the lower curve is $f(x) = x^2 - 4$, but the limits are not known.

To find the limits of the integral, we first find the points of intersection of the two functions by equating the two equations.

$$x^2-4=2x-1$$
$$x2-2x-3=0$$
$$(x+1)(x-3)=0$$
$$x=-1 \quad or \quad x=3$$



The two curves intersect at $x = -1$ and $x = 3$. Thus,

$$\text{Required area} = \int_{-1}^{3} (2x-1)-(x^2-4)\,dx$$

$$= \int_{-1}^{3} -x^2 + 2x + 3\,dx$$

$$= \left[-\frac{x^3}{3} + x^2 + 3x\right]_{-1}^{3}$$

$$= \left[-\frac{3^3}{3} + 3^2 + 3(3)\right] - \left[-\frac{(-1)^3}{3} + (-1)^2 + 3(-1)\right]$$

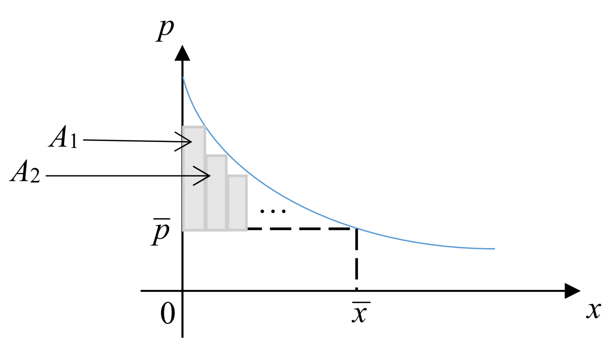
$$= \frac{32}{3} \text{ unit}^2$$

2.2 Consumers' and producers' surplus

In economics, the consumers' surplus is the difference between the total amount that consumers are willing to pay for a product and the total amount that they actually pay for the product. For example, if James is willing to pay \$20 for a toy car and the actual price of the car is \$15, then his surplus is \$5. We will now examine a formula to compute the consumers' surplus in an economy.

Suppose we have a demand function $p = D(x)$ where p is the unit price of a product and x is the quantity demanded. Furthermore, suppose that $\overline{p}$ is the established fixed unit market price and $\overline{x}$ is the corresponding quantity demanded.

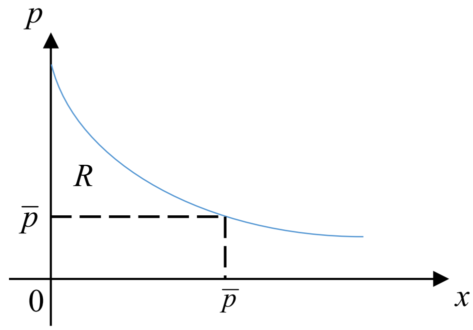
Figure 12.3: Approximating consumers' surplus from the demand function



From Figure 12.3, Divide the interval [0, $\overline{x}$] into n subintervals of equal length Δx . Observe that the area of rectangle A_1 (difference in price times quantity) approximates the consumer surplus for a group of consumers who are willing to pay a unit price higher than the market price for the product for Δx units. Similarly, the area of rectangle A_2 approximates the consumer surplus for a second group of consumers who are willing to pay a different unit price higher than the market price. The total savings to the consumers can thus be approximated by summing all the rectangles.

Following the same process in Chapter 11, as the number of rectangles increases to infinity, the consumers' surplus is given by the area of region R which is bounded above by the demand function and bounded below by the horizontal line $f(x) = \bar{p}$ (Figure 12.4).

Figure 12.4: The consumers' surplus is represented by the area R



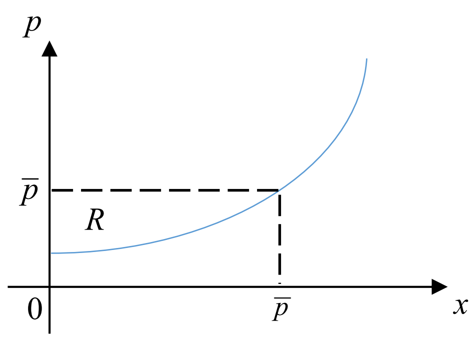
The area of region R can be calculated by evaluating the area under the demand function minus the area of rectangle, with the lower limit at $x = 0$ and the upper limit at $x = \bar{x}$. Therefore, the consumers' surplus is given by

$$CS = \int_0^{\bar{x}} D(x)\,dx - \bar{p}\,\bar{x}$$

where $D(x)$ is the demand function, $\bar{p}$ is the unit market price and $\bar{x}$ is the quantity demanded.

Analogously, the producers' surplus is given by the area of region R which is bounded above by the horizontal line $f(x) = \bar{p}$ and bounded below by the supply function (See Figure 12.5). The area of region R can be calculated by evaluating the area of rectangle minus the area under the supply function, with the lower limit at $x = 0$ and the upper limit at $x = \bar{x}$.

Figure 12.5: The consumers' surplus is represented by the area R



The producers' surplus is given by

$$PS = \bar{p}\,\bar{x} - \int_0^{\bar{x}} S(x)\,dx$$

where $S(x)$ is the supply function, $\bar{p}$ is the unit market price and $\bar{x}$ is the quantity demanded.

Example 12.4 The demand function, $D(x)$, for a set of kitchenware is given by

$$p = -0.1x^2 - x + 40$$

where p is the unit price in dollars and x is the quantity demanded in hundreds of units. The supply function, $S(x)$, for the kitchenware is given by

$$p = 0.1x^2 + 2x + 20$$

where p is the unit price in dollars and x is the quantity that will be supplied in hundreds of units. Find the consumers' and producers' surplus if the market price of the kitchenware is set at the equilibrium price.

Solution

We first determine the equilibrium price and quantity by solving for the intersection point of the demand and supply functions, i.e. by equating the two functions.

$$\begin{aligned}
-0.1x^2 - x + 40 &= 0.1x^2 + 2x + 20 \\
-0.2x^2 - 3x + 20 &= 0 \\
x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-0.2)(20)}}{2(-0.2)} \\
&= -20 \quad or \quad 5
\end{aligned}$$

As x cannot be negative, $x = 5$.

$$\begin{aligned}
p &= -0.1(5)^2 - 5 + 40 \\
&= 32.5
\end{aligned}$$

Hence, $\bar{x} = 5$ and $\bar{p} = 32.5$. Using the formulas for consumers' and producers' surplus, we have

$$\begin{aligned}
\text{Consumers' surplus} &= \int_0^5 -0.1x^2 - x + 40\,dx - (32.5)(5) \\
&= \left[-0.1\left(\frac{x^3}{3}\right) - \frac{x^2}{2} + 40x\right]_0^5 - 162.5 \\
&= 20.8333333 \quad or \quad \$2,083.33 \text{ (} x \text{ is in units of hundreds)}
\end{aligned}$$

$$\begin{aligned}
\text{Producers' surplus} &= (32.5)(5) - \int_0^5 0.1x^2 + 2x + 20\,dx \\
&= 162.5 - \left[0.1\left(\frac{x^3}{3}\right) + x^2 + 20x\right]_0^5 \\
&= 33.3333333 \quad or \quad \$3,333.33 \text{ (} x \text{ is in units of hundreds)}
\end{aligned}$$

3. Other applications of the definite integral

3.1 Future and present value of an income stream

Suppose a company generates a stream of income over a certain period of time. As the income is realized, it is reinvested and earns interest at a fixed rate that is compounded continuously. One method to measure the value of such an income stream is to compute its accumulated value over that particular period. The accumulated value of an income stream is known as the future value of the income stream and can be determined by the formula

$$A = e^{rT}\int_0^T R(t)e^{-rt}\,dt$$

where r = annual interest rate compounded continuously
T = number of years
$R(t)$ = income stream (in dollars per year)

Another method to measure the value of an income stream is to compute its present value. The present value of an income stream is the principal that will yield the same accumulated value as the income stream if the principal is invested today for the same period at the same interest rate. It can be determined by the formula

$$PV = \int_0^T R(t)e^{-rt}\,dt$$

where r = annual interest rate compounded continuously
T = number of years
$R(t)$ = income stream (in dollars per year)

Example 12.5 A consultancy firm generates income at a rate of \$100,000 per year. If the income is reinvested in a business earning interest at a rate of 5% per year compounded continuously, find the (a) total accumulated value of the income stream at the end of 10 years, (b) the present value of the income stream over 10 years.

Solution
(a) The total accumulated value is the future value of the income stream with $R(t)$ = \$100,000, T = 10 and r = 0.05.

$$\begin{aligned}
A &= e^{0.05(10)}\int_0^{10} 100{,}000e^{-0.05t}\,dt \\
&= 100{,}000e^{0.5}\left[\frac{e^{-0.05t}}{-0.05}\right]_0^{10} \qquad \text{[Use integration by substitution]} \\
&= \frac{100{,}000}{-0.05}e^{0.5}\left[e^{-0.05(10)} - e^{-0.05(0)}\right] \\
&= 2{,}000{,}000e^{0.5}\left[0.3934693\right] \\
&= \$1{,}297{,}442.54
\end{aligned}$$

(b) The present value of the income stream can be calculated with $R(t)$ = \$100,000, $T = 10$ and $r = 0.05$.

$$\begin{aligned}
P &= \int_0^{10} 100{,}000e^{-0.05t}\,dt \\
&= 100{,}000\left[\frac{e^{-0.05t}}{-0.05}\right]_0^{10} \qquad \text{[Use integration by substitution]} \\
&= 2{,}000{,}000\left[e^{-0.05(10)} - e^{-0.05(0)}\right] \\
&= \$786{,}938.68
\end{aligned}$$

From Chapter 6, the accumulated value A of a principal P that earns interest at an annual interest rate r compounded continuously is given by the formula $A = Pe^{rt}$. In the example above, we can compute the accumulated value from the present value as well.

$$\begin{aligned}
A &= 786{,}938.68e^{(0.05)(10)} \\
&= \$1{,}297{,}442.54
\end{aligned}$$

3.2 Future and present value of an annuity

Recall that an annuity is a sequence of payments made at regular time intervals. The term of an annuity is the time period in which these payments are made. When payments are made with a fixed compounding period, e.g. monthly, we can apply the formulas in Chapter 6 to find the future and present value of an annuity. However, those formulas do not apply when the annual interest rate is compounded continuously.

Suppose that P is the size of each payment in an annuity and there are m payments in a year. We can consider that these payments constitute an income stream of $R(t) = mp$ dollars per year. Thus, when the interest rate is compounded continuously, we can apply the formulas in the previous section to derive a formula for the future value of an annuity.

$$\begin{aligned}
A &= e^{rT}\int_0^T R(t)e^{-rt}\,dt \\
&= e^{rT}\int_0^T mPe^{-rt}\,dt \qquad [R(t) = mP] \\
&= mPe^{rT}\left[\frac{e^{-rt}}{-r}\right]_0^T \qquad \text{[Integration by substitution]} \\
&= \frac{mPe^{rT}}{-r}\left[e^{-rT} - 1\right] \\
&= \frac{mP}{-r}\left[1 - e^{rT}\right] \\
&= \frac{mP}{r}\left[e^{rT} - 1\right]
\end{aligned}$$

This leads us to the formula for the future value of an annuity.

$$A = \frac{mP}{r}\left[e^{rT} - 1\right]$$

where r = annual interest rate compounded continuously
T = number of years
P = size of each payment in the annuity
m = number of payments in a year

The present value of an annuity follows an analogous definition and derivation.

$$PV = \frac{mP}{r}\left[1 - e^{-rT}\right]$$

where r = annual interest rate compounded continuously
T = number of years
P = size of each payment in the annuity
m = number of payments in a year

Example 12.6 Leslie deposits \$1,500 each month into a retirement account that pays interest at the rate of 4% per year compounded continuously. How much will he have in the account at the end of 20 years?

Solution
The total accumulated value is the future value of the annuity with m = 12, P = \$1,500, T = 20 and r = 0.04.

$$\begin{aligned} A &= \frac{12(1500)}{0.04}\left[e^{0.04(20)} - 1\right] \\ &= \$551,493.42 \end{aligned}$$

Example 12.7 Casey wants to set up a fund in which he will withdraw \$2,000 each month for the next 10 years. The fund earns interest at a rate of 5% per year compounded continuously. How much does he need to set up the fund?

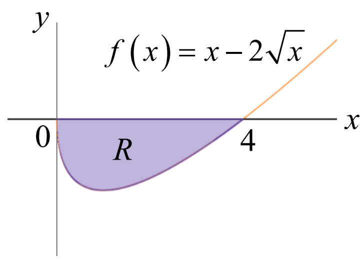
Solution
The amount that Casey needs to set up the fund is given by the present value of the annuity with m = 12, P = \$2,000, T = 10 and r = 0.05.

$$\begin{aligned} PV &= \frac{12(2000)}{0.05}\left[1 - e^{-0.05(10)}\right] \\ &= \$188,865.28 \end{aligned}$$

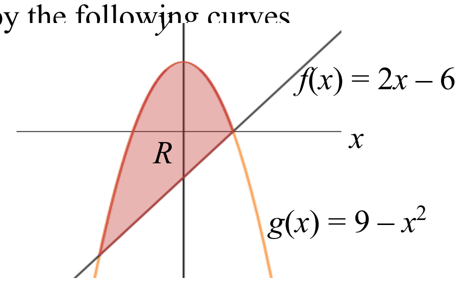
4. Discussion questions

1. Find the area of the shaded region R bounded by the following curves

(a)



(b)



2. Sketch the graph of $f(x) = e^x$. Find the area of the region bounded by the graph of the function f and the lines $y = 0$, $x = 1$ and $x = 2$.

3. The demand function, for a brand of washing detergent is given by $p = -0.01x^2 - 0.2x + 8$ where p is the unit price in dollars and x is the quantity demanded in units of a thousand. The supply function, for the same brand is given by $p = 0.01x^2 + 0.1x + 3$, where p is the unit price in dollars and x is the quantity that will be supplied in units of a thousand. Find the consumers' and producers' surplus if the market price of the washing detergent is set at the equilibrium price.

4. Suppose an investment is expected to generate income at the rate of \$60,000 per year for the next 10 years. Find the future value of the income stream if the prevailing interest rate is 3% per year compounded continuously.

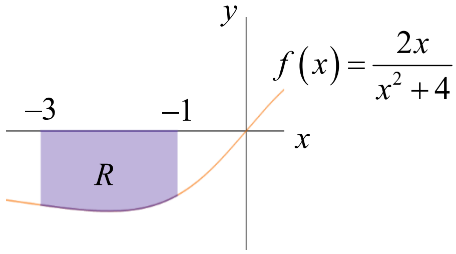
5. Mr and Mrs Lee wants to save up in order to finance their child's tertiary education. To do so, they deposit \$500 each month into a savings account that pays an interest rate of 4% per year compounded continuously. How much will they have in the account in 15 years?

6. Estimate the present value of an annuity if payments are \$2,500 quarterly for 10 years at an interest rate of 12% per year compounded continuously.

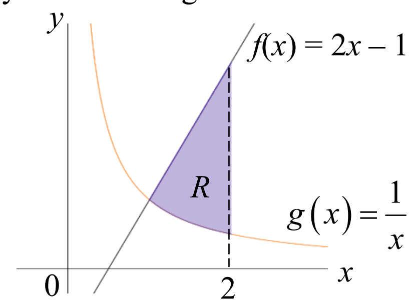
5. Supplementary questions

1. Find the area of the shaded region R bounded by the following curves.

(a)



(b)



2. Find the area of the region bounded by the graph of the function $f(x) = e^{5x-1}$ and the lines $y = 0$, $x = 0$ and $x = 1$.

3. The demand function for a ZZZ mattress is given by

$$p = \frac{600}{x+2}$$

where p is the unit price in dollars and x is the quantity demanded in units of a thousand.

(a) What is the lowest price for which there is no demand?

(b) Find the consumer's surplus if the equilibrium price is \$200 per unit.

4. The supply function for a particular model of electric toothbrush is given by

$$p = \sqrt{36 + 1.8x}$$

where p is the unit price in dollars and x is the quantity supplied in units of a hundred. Find the producers' surplus if the market price is set at \$9 per unit.

5. Michael bought a franchise for a hardware store that is expected to generate income at the rate of \$80,000 per year for the next 6 years. Find the present value of the franchise if the interest rate is 2% per year compounded continuously.

6. Joyce intends to set up a fund in which she will withdraw \$500 each month for the next 20 years. If the fund earns interest at a rate of 4% per year compounded continuously, how much does she need to set up the fund?

Answers

Chapter 1 Discussion

1(a)	$\dfrac{1}{5a^4}$	(b)	$\dfrac{16b}{a}$
(c)	$5p^6q^8$	(d)	$\dfrac{5u^9}{v^8}$
2(a)	$3x^5+3x^3-2x^2-2$	(b)	$49x^6-14x^4+x^2$
(c)	$24x+24$	(d)	$-5x^2-4$
3(a)	$(5x-12y)(a+2)$	(b)	$25x(x+7)(x-7)$
(c)	$(4y-1)(16y^2+4y+1)$	(d)	$(x-2)(2x+1)$
4(a)	$2x+8$	(b)	$\dfrac{x-3}{x-1}$
(c)	$\dfrac{6x+44}{(x+6)(x-6)(x-2)}$	(d)	$\dfrac{x^2+4x+4}{x+5}$
5(a)	$10u^3v^2w^5$	(b)	$\dfrac{2\sqrt[3]{x^2}}{x}$
(c)	$\dfrac{24+16\sqrt{x}}{9-4x}$	(d)	$\dfrac{x\sqrt{x+1}}{x+1}$

Chapter 1 Supplementary

1(a)	$\dfrac{9b^{16}}{a^4}$	(b)	$\dfrac{64}{p^8q^3r^2}$
(c)	$\dfrac{25u^{12}}{64v^{24}}$	(d)	$\dfrac{5}{8x^2y^3}$
2(a)	$13a^2-19a$	(b)	$0.24p^2-0.78pq-2.52q^2$
(c)	$17u^2-2uv-8v^2$	(d)	$5x^3-10x^2-50x$
3(a)	$(x+4y)(2a+b)$	(b)	$(4u^2+9v^2)(2u+3v)(2u-3v)$
(c)	$(y+10)(y-8)$	(d)	$(z-3)(5z+1)$
4(a)	$3r-5$	(b)	$\dfrac{u+3}{u-2}$
(c)	$\dfrac{x+20}{(x-4)^2(x+4)}$	(d)	$\dfrac{-x^2-x+3}{(x+5)(x-5)}$
5(a)	m^5np^6	(b)	$\dfrac{6\sqrt{x+1}}{x+1}$
(c)	$\dfrac{3\sqrt{x}+6}{x-4}$	(d)	$\dfrac{(6x-5)\sqrt{x-1}}{5x-5}$

Chapter 2 Discussion

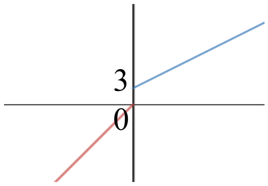
1(a)	14	(b)	20
(c)	1.25	(d)	5 or 2
(e)	11.2 or –31.2	(f)	–0.382 or –2.62
2	3.71 or 21.5 seconds		
3(a)	$x \geq 6$	(b)	$2 < x < 16$
(c)	$x \leq -3$ or $x \geq 5$	(d)	$-7 < x \leq \frac{1}{2}$
(e)	$-\frac{3}{5} \leq x \leq 5$	(f)	$x > \frac{16}{3}$ or $x < -6$
4	$\frac{376}{5} \leq F \leq 95$		
5(a)	102; 795	(b)	$\frac{1}{9}$; $\frac{3280}{9}$
6	23^{rd} day	7	680,244 people

Chapter 2 Supplementary

1(a)	–8	(b)	72
(c)	–1	(d)	1 or 0.5
(e)	0.531 or –7.53	(f)	0.271 or –2.77
2	2 or 0.5 hours		
3(a)	$x < 5$	(b)	$-\frac{1}{2} < x < -\frac{1}{9}$
(c)	$-8 < x < 8$	(d)	$x \leq -\frac{5}{2}$ or $x > 2$
(e)	$-\frac{47}{9} \leq x \leq 5$	(f)	$x > 5$ or $x < \frac{-13}{7}$
4	Smallest diameter = 0.09 inches. Largest diameter = 0.11 inches		
5(a)	–3; 132	(b)	–39,366; –29524
6(a)	Company B	(b)	Company A

Chapter 3 Discussion

1(a)	$-\frac{6}{5}$	(b)	$\frac{1}{4}$
2(a)	$x = 5$	(b)	$y = 2x + 5$
3	$y = 4x + 2$	4	$y = -1.5x + 3.5$
5	4; 2; $\frac{4}{3}$		
6(a)	$(-\infty, \infty)$	(b)	$(-\infty, 5]$
(c)	$(-\infty, -1) \cup (-1, 3) \cup (3, \infty)$	(d)	$(-\infty, -2) \cup (-2, 1]$

7 

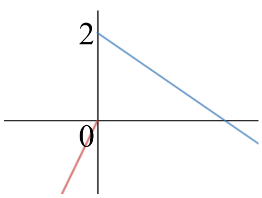
Range = (–∞, 0] ∪ (3, ∞)

8 x^2+3x; x^2+x-2; $\left(x^2+2x-1\right)\left(x+1\right)$; $\dfrac{x^2+2x-1}{x+1}$

9 $g\circ f\left(x\right)=\sqrt{x^2+1}$; $f\circ g\left(x\right)=x+1$

Chapter 3 Supplementary

1(a)	2	(b)	0
2(a)	$y = -2x + 11$	(b)	$x = 3$
3	$y = -0.5x + 1$	4	$3y = 8x + 7$
5	81; $\dfrac{25}{16}$		
6(a)	(–∞, ∞)	(b)	(–∞, 0.5]
(c)	(–∞, –1) ∪ (–1, ∞)	(d)	(–∞, –1) ∪ (1, ∞)

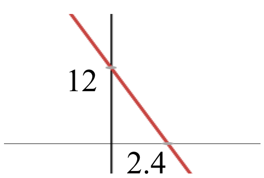
7 

Range = (–∞, 2)

8 $-x^2+7$; $-3x^2+15$; $\left(-2x^2+11\right)\left(x^2-4\right)$; $\dfrac{-2x^2+11}{x^2-4}$

9 $g\circ f\left(x\right)=2\left(3^{x-1}\right)+1$; $f\circ g\left(x\right)=3^{2x}$

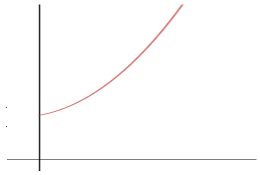
Chapter 4 Discussion

1(a) 

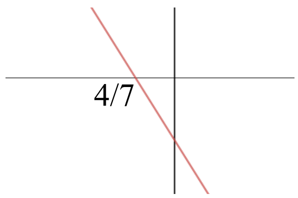
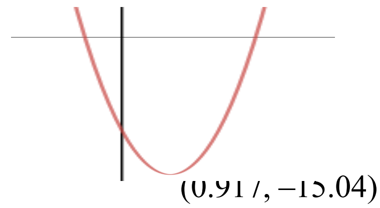
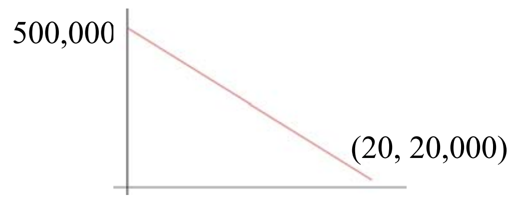
(b) 

(3.5, –2.25)

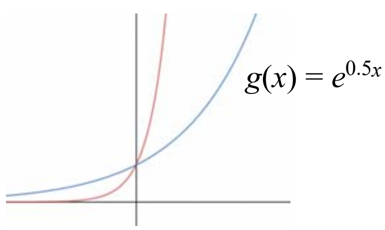
2(a)	(3, –13)	(b)	(–1, 0) and (6, 77)
3(a)	\$60,000 per year	(b)	$V(t) = -60{,}000t + 600{,}000$
(c)	\$360,000		
4(a)	$C(x) = 8x + 480{,}000$; $R(x) = 14x$; $P(x) = 6x - 48{,}000$		
(b)	8000; \$112,000	(c)	\$12,000

5(a)		(b)	\$20
(c)	\$15	6	5000; \$65
7(a)	0.06 million terabytes	(b)	12.732 million terabytes

Chapter 4 Supplementary

1(a)		(b)	
2(a)	(2, –8) and (–2, 12)	(b)	(–3.39, 2.37) and (7.19, –2.28)
3(a)	$V(t) = -24{,}000t + 500{,}000$	(b)	
(c)	\$24,000 per year	(d)	\$236,000
4(a)	$C(x) = 4x + 30{,}000$; $R(x) = 10x$; $P(x) = 6x - 30{,}000$		
(b)	5000; \$50,000	(c)	7000
5(a)	\$108	(b)	12000
(c)	8000; \$80		
6	51,487	(b)	No.

Chapter 5 Discussion

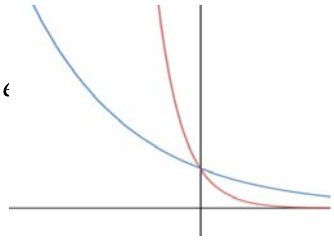
1(a)	4	(b)	2
(c)	$\frac{15}{7}$	(d)	1 or 2
2			
3(a)	$\log x + 6 \log (x + 1)$	(b)	$x - \ln (1 + e^x)$
4(a)	25	(b)	10
5(a)	8.05	(b)	0.999
(c)	73.7	(d)	36.7
6(a)	666 million	(b)	926.755 million

7	11.9 minutes		
8(a)	10	(b)	88

Chapter 5 Supplementary

1(a)	–6	(b)	–17
(c)	$\frac{5}{9}$	(d)	1

2 $g(x) =$



3(a)	$\frac{1}{2}\log(x-1)-3$	(b)	$2x-\frac{1}{2}\ln x$
4(a)	343	(b)	6
5(a)	7.54	(b)	–3.52
(c)	21.9	(d)	4.66
6(a)	2540	(b)	29.9 months
(c)	4000		
7	747 people		

Chapter 6 Discussion

1(a)	\$300	(b)	\$11,255.48; \$11,400.79; \$11,414.71
2	9.308%; 9.381%	3	\$31,086.07
4(a)	\$3,180	(b)	\$73,902.04
5(a)	\$232,923.42	(b)	\$43,698.36
6(a)	\$836.44	(b)	\$169.77
7(a)	\$1,201.55	(b)	\$1,569,142.11

Chapter 6 Supplementary

1(a)	\$5150	(b)	\$26,966.97
2(a)	6.168%	(b)	\$12,390.48
3(a)	\$984,905.99	(b)	\$811,152.89
4(a)	\$1,157.81	(b)	\$2,740.30
5	\$76,386.45	6	\$31,666.77
7	\$1,027.92		

Chapter 7 Discussion

1(a)	5	(b)	$\frac{1}{2}$
(c)	$\frac{1}{6}$	(d)	14
(e)	1	(f)	2
2	No.		

3(a)	$(-\infty, \infty)$	(b)	$(-\infty, -5) \cup (-5, -2) \cup (-2, \infty)$
4	2; Yes, $m = 2$ from $y = mx + b$		
5(a)	$5x^4$	(b)	$-\frac{3}{x^2}$
(c)	$\frac{2}{\sqrt{x}} - 1$	(d)	$-\frac{21}{x^4} - \frac{1}{2x\sqrt{x}}$
(e)	$2t - 6$	(f)	$4w - \frac{3}{w^2}$
6(a)	$30x^2$	(b)	30
(c)	$y = 30x - 20$		

Chapter 7 Supplementary

1(a)	$\frac{2}{5}$	(b)	25
(c)	$-\frac{1}{10}$	(d)	0
(e)	$\frac{3}{5}$	(f)	∞
2	Yes; 1		
3(a)	$(-\infty, \infty)$	(b)	$(-\infty, \infty)$
4	$2x - 2$; 1		
5(a)	$0.02x - 0.5$	(b)	$\frac{3}{2}\sqrt{x}$
(c)	$2x$	(d)	$1 - \frac{2}{x^3}$
6(a)	$3x^2 - 3$	(b)	1 or -1

Chapter 8 Discussion

1(a)	$8x^3 + 3x^2 - 2$	(b)	$-\frac{5}{(x-2)^2}$
(c)	$40x\left(x^2 + 15\right)^{19}$	(d)	$\frac{(3x+1)\sqrt{2x+1}}{2x+1}$
2(a)	$7e^{7x+1}$	(b)	$\frac{1}{x}$
(c)	$2e^{2x+3} - \frac{1}{x+1}$	(d)	$2xe^x + 3e^x$
(e)	$1 + \ln x$	(f)	$\frac{7-5x}{e^x}$
3(a)	$4e^{4x-1}$	(b)	$4e^3$
4	$\overline{C}'(x) = -\frac{500,000}{x^2}$		
5(a)	$R(x) = 600x - 0.05x^2$; $P(x) = -0.000002x^3 - 0.02x^2 + 200x - 80000$		

(b) $C'(x) = 0.000006x^2 - 0.06x + 400$; $R'(x) = 600 - 0.1x$; $P'(x) = -0.000006x^2 - 0.04x + 200$

(c) \$96; Profit realised from the 2001st unit

6(a) Increasing on $(-\infty, \infty)$

(b) Increasing on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$, decreasing on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

7(a) (2, –4) relative minimum

(b) (2, 1) relative minimum, (0, 5) relative maximum

Chapter 8 Supplementary

1(a) $-5x^4 + 3x^2 + 14x$

(b) $-\frac{29}{(5x+2)^2}$

(c) $(72x - 24)(1 - 2x + 3x^2)^{11}$

(d) $\frac{(1-6x)\sqrt{1-4x}}{1-4x}$

2(a) $e^{5x+1}\left(\frac{1}{x} + 5\ln x\right)$

(b) $\frac{-3x^2 + 6x + 1}{e^x}$

(c) $\left(8e^{2x} + \frac{12}{x}\right)\left(e^{2x} + 3\ln x\right)^3$

(d) $\frac{30x}{3x^2 - 1}\left[\ln\left(3x^2 - 1\right)\right]^4$

3 $\frac{50 - 0.5x^2}{\left(0.01x^2 + 1\right)^2}$

4(a) $C'(x) = 0.000006x^2 - 0.04x + 120$; $R'(x) = 190 - 0.012x$; $P'(x) = -0.000006x^2 + 0.028x + 70$

(b) \$102; Profit realised from the 2001st unit

5(a) Decreasing on $(-\infty, 0)$, increasing on $(0, \infty)$

(b) Increasing on $(-\infty, -6)$ and $(0, \infty)$, decreasing on $(-6, 0)$

6(a) (1, 14) relative minimum

(b) (9, 977) relative minimum

Chapter 9 Discussion

1(a) $42x^5 - 72x^2$; $210x^4 - 144x$

(b) $9e^{3x-1}$; $27e^{3x-1}$

2(a) Inflection at (1, –20), concave downwards on $(-\infty, 1)$, concave upwards on $(1, \infty)$

(b) Inflection at (0, 6) and (1, 5), concave upwards on $(-\infty, 0)$ and $(1, \infty)$, concave downwards on (0, 1)

3(a) $\left(-\frac{3}{4}, \frac{47}{8}\right)$ relative minimum

3(b) (3, 6) relative minimum, (–3, –6) relative maximum

4(a) absolute minimum = –1, absolute maximum = 199

(b) absolute minimum = 1, absolute maximum = 1.901

5 \$3,600

6 46

7 \$17,600

Chapter 9 Supplementary

1(a)	$180x^8 + 18x$; $1440x^7 + 18$	(b)	$-\frac{25}{(5x+2)^2}$; $\frac{250}{(5x+2)^3}$
2(a)	Sales increase over time from 1993 to 2000.		
(b)	Sales grew at a slower rate at the initial period and grew at a faster rate over time.		
3(a)	(1, 3) relative minimum	(b)	(1, 0.368) relative minimum
4	\$53.12 per square foot	5	2118
6(a)	1000	(b)	1000
(c)	They are the same. There are two ways to calculate the level of production that results in the lowest average cost.		
7	467		

Chapter 10 Discussion

1(a)	$\frac{1}{11}x^{11} + c$	(b)	$7x - \frac{2}{3}x^{\frac{3}{2}} + c$
(c)	$-\frac{4}{x^2} + 2x^{\frac{1}{2}} + c$	(d)	$\frac{1}{3}x^3 + \frac{1}{2}x^2 + c$
(e)	$4e^x - 3x + c$	(f)	$\ln x + \frac{1}{x} + c$
2	$f(x) = \frac{1}{2}x^4 + \frac{1}{2}x^2 - 5x + 7$		
3(a)	$f(t) = -1.34t^3 + 30t^2 + 240t + 3000$	(b)	\$6,613.14
4	\$2,022		
5(a)	$\frac{1}{5}\left(2x^3+3\right)^5 + c$	(b)	$\frac{2}{3}\left(2x^2-1\right)^{\frac{3}{2}} + c$
(c)	$\frac{1}{3}\ln\left(3x^2-1\right) + c$	(d)	$2\ln\left(1+e^x\right) + c$
(e)	$100e^{0.01x} + c$	(f)	$\frac{1}{2}\left(\ln 5x\right)^2 + c$
6	38.2 million		

Chapter 10 Supplementary

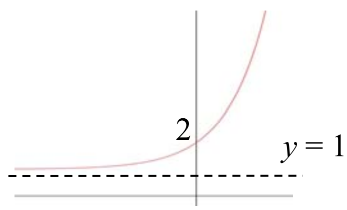
1(a)	$2e^x - \ln x + c$	(b)	$\frac{1}{3}x^3 + x^2 + x + c$
(c)	$\frac{1}{3}x^3 - 49x + c$	(d)	$\frac{5}{2}x^2 - x + 3\ln x + c$
2	$f(x) = e^x + \frac{1}{2}x^2 - 2x + 2$	3	11,148 subscribers
4(a)	5000	(b)	\$60,000
5(a)	$\frac{1}{4}\left(x^3-2x\right)^4 + c$	(b)	$-e^{2-x} + c$
(c)	$e^{x^2} + c$	(d)	$2\ln\left(\ln x\right) + c$

6	99.5 words per minute		

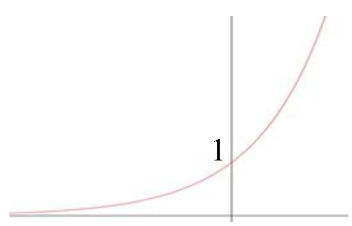
Chapter 11 Discussion

1(a)	20.5	(b)	20.64
2(a)	–18	(b)	3.92
(c)	2.44	(d)	14.6
3	33.07		
4(a)	8191.875	(b)	1.236
(c)	51.9	(d)	0.568
5	1.22	6	31.7
7(a)	$\frac{47}{3}$	(b)	$\frac{1}{6}\ln 3$
8(a)	148.504 million	(b)	140.12 million

Chapter 11 Supplementary

1(a)	6.13	(b)	6.35
2(a)	2	(b)	44
(c)	5.64	(d)	–2.35
3(a)		(b)	152
4(a)	323.75	(b)	$\frac{2}{3}$
(c)	72.8	(d)	0.981
5	209.8		
6(a)	\$5,800	(b)	\$900
7(a)	3	(b)	128
8(a)	0.16 million	(b)	2.066 million

Chapter 12 Discussion

1(a)	$\frac{8}{3}$	(b)	$\frac{256}{3}$
2	 Area ≈ 4.67	3	\$16,666.67; \$11,666.67
4	\$699,716.59	5	\$123,317.82
6	\$58,233.82		

Chapter 12 Supplementary

1(a)	0.956	(b)	1.31
2	10.8		
3(a)	300	(b)	$43,279.06
4	$3,500	5	$452,318.25
6	$82,600.66		